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G. Markowsky
B. K. Rosen

Bases for Chain-complete Posets*

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Abstract: Various authors (especially Scott, Egli, and Constable) have introduced concepts of "basis" for various classes of partially ordered sets (posets). This paper studies a basis concept directly analogous to the concept of a basis for a vector space. The new basis concept includes that of Egli and Constable as a special case, and one of their theorems is a corollary of our results. This paper also summarizes some previously reported but little known results of wide utility. For example, if every linearly ordered subset (chain) in a poset has a least upper bound (supremum), so does every directed subset.

Given posets P and Q, it is often useful to construct maps $g:P \rightarrow Q$ that are *chain-continuous*: supremums of nonempty chains are preserved. Chain-continuity is analogous to topological continuity and is generally much more difficult to verify than *isotonicity*: the preservation of the order relation. This paper introduces the concept of an *extension basis*: a subset B of P such that any isotone $f:B \rightarrow Q$ has a unique chain-continuous extension $g:P \rightarrow Q$. Two characterizations of the chain-complete posets that have extension bases are obtained. These results are then applied to the problem of constructing an extension basis for the poset $[P \rightarrow Q]$ of chain-continuous maps from P to Q, given extension bases for P and Q. This is not always possible, but it becomes possible when a mild (and independently motivated) restriction is imposed on either P or Q. A lattice structure is not needed.

1. Introduction

Scott [1] proposed that lattice theory should play a fundamental role in the theory of computing. Various aspects of lattice theory with computer science motivations have been studied by many authors, among them Goguen, Thatcher, Wagner, and Wright [2, 3], Markowsky [4, 5], Plotkin [6], and Scott [7, 8]. Space does not permit a full survey of the computer science applications of lattice theory. The diversity of applications is illustrated by the work of Cadiou and Levy [9], Hitchcock and Park [10], Lewis and Rosen [11], Rosen [12], and Vuillemin [13]. Further references can be found in the works cited, especially [2].

Much of the applied lattice theory in computer science does not use lattices! Where Scott would recommend complete lattices [1] or continuous lattices [7], a more general class of mathematical structures has been used. Following [4, 5], we call members of this class *chain-complete posets*. This class is used in [11, 12]. The slightly larger class of *ω-chain-complete posets* is used in [9, 13]. Definitions are in Section 2.

Chain-complete posets have numerous technical advantages over complete lattices for computing applications. Certain universal constructions are possible with chain-complete posets but impossible with complete lattices [5]. One conjectured disadvantage is well known in the folklore of this subject: if P, Q are chain-complete posets with "effectively given bases," then the poset of continuous maps $[P \rightarrow Q]$ may not have an effectively given basis. Clearly, the truth of this conjecture depends on the precise definition chosen for the basis concept sketched in a lattice oriented manner by Scott [1, Sec. 4]. One such definition is that of "recursive bases" proposed by Egli and Constable [14, Sec. II.2] who show that $[P \rightarrow Q]$ does have a recursive basis whenever P and Q have recursive bases. Expressed in terms of different definitions, Vuillemin's Lemma 2 [15, Chap. III] is equivalent to this result. In Section 5 of this paper, we derive this result as a special case of more general theorems dealing with separate concepts of basis and of recursive listability that have independent mathematical motivations.

Section 2 begins with basic definitions and facts about chain-complete and ω-chain-complete posets. We introduce a concept of compactness inspired by lattice theory and a universal construction inspired by Theorem 4 of [15]. The *basis completion* $\overline{P}$ of a poset P is a chain-complete poset such that isotone maps from P to Q correspond to chain-continuous maps from $\overline{P}$ to Q.

Section 3 defines an *extension basis* for P to be a subset B of P such that any isotone $f:B \rightarrow Q$ has a unique chain-continuous extension $g:P \rightarrow Q$. Theorem 3.2 shows that B is an extension basis for P iff P is isomorphic to $\bar{B}$ in a certain natural way. Theorem 3.3 shows that P has an extension basis iff every member x of P is the supremum of a directed set B_x consisting of all compact c with

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$c \leq x$. This characterization is helpful in relating extension bases to the narrower basis concepts studied by Egli and Constable [14] and by Vuillemin [15].

Section 4 deals with the poset $[P \rightarrow Q]$ of chain-continuous maps from P to Q. Given extension bases for P and Q, it need not be possible to construct an extension basis for $[P \rightarrow Q]$. Suppose, however, that either P or Q has *bounded joins*: every finite subset with an upper bound has a supremum. Theorem 4.5 shows that then $[P \rightarrow Q]$ does have an extension basis. The construction generalizes the one used in [14].

Section 5 defines a *recursive listing* for a subset B of P to be a map from the nonnegative integers onto B with appropriate decidability properties. A chain-complete poset with an extension basis B and a recursive listing for B is *recursively based*; this is our formalization of Scott's "effectively given basis" [1, Sec. 4] when chain-complete posets that need not be lattices are considered. Theorem 5.3 shows that $[P \rightarrow Q]$ is recursively based whenever P, Q are recursively based and Q has bounded joins. Theorem 5.6 suggests that stronger results would require the use of oracles. Corollary 5.8 relates this work to [14, 15].

Various common notations from lattice-theoretic computer science are used here: $[P \rightarrow Q]$, $\perp$, **if** $\cdots$ **then** $\cdots$ **else** $\cdots$, and so on. In general this paper is consistent with the notation and terminology of standard works on lattice theory [16, 17, 18], with a few clearly motivated departures like the use of $\perp$. To avoid superfluous parentheses, the value of a function f at an argument x is just fx rather than $f(x)$. The image set $\{fx | x \in C\}$ is just fC rather than $f(C)$ or $f[C]$.

2. Chain-complete posets and compactness

A *poset* is a nonempty set P together with a partial order $\leq$ on P: the relation $\leq$ must be reflexive, antisymmetric, and transitive. An *upper bound* for $S \subseteq P$ is any x in P such that $a \leq x$ for all a in S. A least upper bound or *supremum* for $S \subseteq P$ is any upper bound x for S such that $x \leq y$ for all upper bounds y. In general, S may not have upper bounds and may not have a supremum even if it has upper bounds. The supremum of S, if any, is denoted sup S.

For ease of reference we repeat some definitions from [4].

Definition 2.1 Let P be a poset and $S \subseteq P$. Then S is a *chain* iff, for all a, b in S, either $a \leq b$ or $b \leq a$. On the other hand, S is *directed* iff every finite subset of S has an upper bound in S. The poset P is (ω-)*chain-complete* iff every (countable) chain in P has a supremum.

The empty set $\varnothing$ is a chain but is not a directed set. Nonempty chains are directed. If P is chain-complete, then sup $\varnothing$ is a least element and is denoted $\perp$. It is easy to construct posets that are ω-chain-complete but not chain-complete. Chain-completeness is more convenient than ω-chain-completeness and seems to entail no significant loss of generality. The ω-chain-complete posets that have actually arisen in computer science are also chain-complete, and countability has never been exploited in a completeness verification.

Definition 2.2 Let P and Q be posets and $f: P \rightarrow Q$ be a map. Then f is *isotone* iff, for all x, y in P,

$$x \leq y \text{ implies } fx \leq fy. \tag{1}$$

The map f is (ω-)*chain-continuous* iff, for each nonempty (countable) chain C with a supremum in P, the image set $fC \subseteq Q$ has a supremum in Q and

$$f(\sup_P C) = \sup_Q (fC). \tag{2}$$

Isotone maps have sometimes been called "monotone" or "monotonic." Chain-continuous maps have sometimes been called "continuous." The qualifier "chain" is retained here, but the following lemma shows that it could be omitted in the future without serious ambiguity.

Lemma 2.3 Any (countable) directed subset of a (ω-) chain-complete poset has a supremum. Moreover, let P and Q be (ω-)chain-complete posets, and let $f: P \rightarrow Q$ be (ω-)chain-continuous. Then, for every (countable) directed $D \subseteq P$,

$$f(\sup_P D) = \sup_Q (fD). \tag{3}$$

Proof See Corollary 2 and Corollary 3 in [4] and note that countability can be imposed throughout. □

Theorem 1 in [4] implies Iwamura's Lemma [19], a very useful fact about directed sets. We state the lemma here for ease of reference.

Lemma 2.4 Let the subsets of a poset P be partially ordered by set inclusion. Any infinite directed $D \subseteq P$ is the union of a nonempty chain $\mathscr{C}$ of directed sets that have cardinalities less than that of D. □

Definition 2.5 A member x of a poset P is (ω-)*chain-irreducible* iff, for every nonempty (countable) chain $C \subseteq P$,

$$x = \sup C \text{ implies } x \in C. \tag{1}$$

It is (ω-)*chain-compact* iff, for every nonempty (countable) chain $C \subseteq P$,

$$x \leq \sup C \text{ implies } (\exists y \in C)(x \leq y). \tag{2}$$

Note that (ω-)chain-compactness implies (ω-)chain-irreducibility. The converse fails, as can be seen in Example 3.4.

Lemma 2.6 Let x be a member of a (ω-)chain-complete poset P, and let D be a (countable) directed subset. If x is (ω-)chain-irreducible, then

$$x = \sup D \text{ implies } x \in D. \quad (1)$$

If x is (ω-)chain-compact, then

$$x \leq \sup D \text{ implies } (\exists y \in D)(x \leq y). \quad (2)$$

Proof We use induction on the cardinality of D. For finite D, (1) and (2) are easily checked. Now suppose D is infinite and (1), (2) hold for all directed sets of smaller cardinality. Let $\mathscr{C}$ be a chain of such sets from Lemma 2.4 with

$$D = \bigcup_{A \in \mathscr{C}} A,$$

so that each A has a supremum by Lemma 2.3 and

$$\sup D = \sup \{\sup A | A \in \mathscr{C}\}.$$

Now (1), (2) for D follow from (1), (2) for each A in $\mathscr{C}$ and the fact that $\{\sup A | A \in \mathscr{C}\}$ is a nonempty chain. □

If P is a complete lattice, then the above lemma may be used to show that chain-compactness agrees with the usual notion of compactness in lattice theory [16, p. 168; 17, p. 13; 18, p. 93]. We therefore omit the qualifier "chain." Compact elements have sometimes been called "finite" or "isolated."

The following theorem can be derived from Theorem 4 of [5], but a direct proof is more convenient here.

Theorem 2.7 Let P be a poset with a least element $\perp$. There is a chain-complete poset $\overline{P}$ (called the *basis completion* of P) and a map $i:P \rightarrow \overline{P}$ (called the *natural embedding*) with the following properties. First, for any chain-complete poset Q and isotone map $f:P \rightarrow Q$, there is a unique $g:\overline{P} \rightarrow Q$ such that

$$g \text{ is continuous and } g \circ i = f. \quad (1)$$

Second, all x, y in P have

$$x \leq y \text{ in } P \text{ iff } ix \leq iy \text{ in } \overline{P}. \quad (2)$$

Third, for any ξ in $\overline{P}$, the following conditions are equivalent:

$$\xi \text{ is chain-irreducible;} \quad (3)$$

$$\xi \text{ is compact;} \quad (4)$$

$$\xi = ix \text{ for some } x \text{ in } P. \quad (5)$$

Fourth, for any ξ in $\overline{P}$, the set

$$J_\xi = \{ia | a \in P \text{ and } ia \leq \xi\} \text{ is directed} \quad (6)$$

and has

$$\xi = \sup J_\xi. \quad (7)$$

Proof Let $\overline{P}$ be the set of all directed $D \subseteq P$ such that $x \leq y$ and $y \in D$ imply $x \in D$. Partially order $\overline{P}$ by set inclusion. Then $\overline{P}$ is chain-complete, and the supremum of a directed set is just its union as a family of sets. Define i by

$$ix = \{a \in P | a \leq x\}$$

to derive (2) immediately. Now consider J_ξ in (6). If ia and ib are in J_ξ, then a and b are in $\xi \subseteq P$ and so some c in ξ has $a \leq c$ and $b \leq c$. Therefore ic in J_ξ has $ia \leq ic$ and $ib \leq ic$. This proves (6). For (7), note that

$$\xi = \bigcup_{a \in \xi} ia = \bigcup_{ia \in J_\xi} ia = \sup J_\xi.$$

We prove the extension property. Given Q and f, note that fD for any $D \in \overline{P}$ is directed in Q. By Lemma 2.3 for directed subsets of Q, the map $g:\overline{P} \rightarrow Q$ with

$$gD = \sup_Q fD$$

is well defined. It is easy to check that g satisfies (1). If h does also, then

$$hD = h(\sup_{\overline{P}} J_D) = \sup_Q (hJ_D)$$
$$= \sup_Q \{(h \circ i)x | x \in D\}$$
$$= \sup_Q fD = gD.$$

We prove that (3) through (5) are equivalent. Clearly (4) implies (3). To show that (5) implies (4), suppose (5) and consider any nonempty chain $\mathscr{C} \subseteq \overline{P}$ with $\xi \leq \sup \mathscr{C}$. For $\xi = ix$ some A in $\mathscr{C}$ has $x \in A$ and hence $\xi \leq A$. To show that (3) implies (5), suppose (3) and apply Lemma 2.6(1) to (6) and (7). □

To derive the analogous result for only ω-completeness and ω-continuity by the same argument, we would need to add the hypothesis that P is countable. For comparisons with works such as [14] that explicitly assume only ω-completeness, it is helpful to know that ω-completeness implies completeness under some frequently occurring conditions.

Lemma 2.8 Let P be an ω-chain-complete poset and $B \subseteq P$ be countable. Suppose that for each $x \in P$, B_x is directed and $x = \sup B_x$, where $B_x = \{b \in B | b \leq x\}$. Then P is chain-complete.

Proof For any chain $C \subseteq P$, we let $A = \bigcup_{x \in C} B_x$. It is easy to see that A is directed. By Lemma 2.3 and countability of $A \subseteq B$, A has a supremum. Clearly, $\sup A = \sup C$. □

Corollary 2.9 Let P be an ω-chain-complete poset and $B \subseteq P$ be countable. Suppose that each member of B is ω-compact and each x in P has $x = \sup E_x$ for some directed $E_x \subseteq B$. Then P is chain-complete.

Proof Let B_x be as above. Clearly $x = \sup E_x \leq \sup B_x \leq x$. We claim that B_x is directed. Let a, $b \in B_x$. Because a,

$b \leq \sup E_x$ there exist $a', b' \in E_x$ such that $a \leq a', b \leq b'$. However, E_x is directed. Thus there exists $c \in E_x$ such that $a', b' \leq c$. Because $E_x \subseteq B_x$, $c \in B_x$ and B_x is directed. □

3. Extension bases

An extension basis lets us obtain continuous maps from isotone maps. Our definition is a direct analog of the following characterization of a basis for a vector space. A subset B of a vector space V is a basis iff, for every vector space W and map $f:B \to W$, there is a unique linear extension $g:V \to W$.

Definition 3.1 A subset B of a chain-complete poset P is an *extension basis* iff, for every chain-complete poset Q and isotone map $f:B \to Q$, there is a unique chain-continuous extension $g:P \to Q$ of f.

Every finite poset with a least element has itself as extension basis. The following two theorems characterize the chain-complete posets that have extension bases.

Theorem 3.2 Let P be a chain-complete poset and $B \subseteq P$. Then B is an extension basis iff there is an isomorphism $h:P \to \overline{B}$ that extends the natural embedding $i:B \to \overline{B}$.

Proof Let $i:B \to \overline{B}$ be the natural embedding in Theorem 2.7, and let $j:B \to P$ be the inclusion map from B into P. By Theorem 2.7(1) there is a unique $g:\overline{B} \to P$ such that

$$g \text{ is continuous and } g \circ i = j. \tag{1}$$

Now suppose B is an extension basis, so that there is also a unique $f:P \to \overline{B}$ such that

$$f \text{ is continuous and } f \circ j = i. \tag{2}$$

By (1) and (2), $(g \circ f):P \to P$ with $(g \circ f)$ continuous and $(g \circ f) \circ j = j$. By uniqueness in Definition 3.1 with j in the role of f there, $(g \circ f)$ is the identity map. Similarly, by uniqueness in Theorem 2.7(1) with i in the role of f there, $(f \circ g):\overline{B} \to \overline{B}$ is the identity map. Therefore P is isomorphic to $\overline{B}$.

Now suppose $h:P \to \overline{B}$ is an isomorphism that extends i. Given isotone $f:B \to Q$, there is a unique $\bar{g}:\overline{B} \to Q$ such that $\bar{g}$ is continuous and $\bar{g} \circ i = f$. Then $g = \bar{g} \circ h$ is a continuous extension of f. Uniqueness follows from uniqueness of $\bar{g}$. □

Theorem 3.3 Let P be a chain-complete poset, and let B be the set of all compact members of P. Then P has an extension basis iff, for each x in P, the set

$$B_x = \{b \in B \mid b \leq x\} \text{ is directed} \tag{1}$$

and has

$$x = \sup B_x. \tag{2}$$

In that case any extension basis B' has

$$B' = B = \{c \in P \mid c \text{ is chain-irreducible}\}. \tag{3}$$

Proof Suppose B' is an extension basis, so that P is isomorphic to $\overline{B'}$ by Theorem 3.2. From (3)-(5) of Theorem 2.7 we can derive (3). To derive (1) and (2) from (6) and (7) of Theorem 2.7, it will suffice to show that the reciprocal isomorphisms $g:\overline{B} \to P$ and $f:P \to \overline{B}$ have $B_x = gJ_{fx}$. Indeed,

$$\begin{aligned} B_x &= \{b \in B \mid b \leq x\} \\ &= \{gfb \mid b \in B \;\&\; fb \leq fx\} \\ &= \{gib \mid ib \leq fx\} = g\, J_{fx}. \end{aligned}$$

Now suppose (1) and (2) for all x in P. We claim that B is an extension basis. Let $f:B \to Q$ be isotone. Define $g:P \to Q$ by

$$gx = \sup_Q f\, B_x, \tag{4}$$

as is possible because $f\, B_x$ is directed. Consider any nonempty chain $C \subseteq P$. For $y = \sup_P C$ we calculate that

$$\begin{aligned} g \sup_P C &= \sup_Q f\, B_y \\ &= \sup_Q \{fb \mid b \in B \;\&\; b \leq y\} \\ &= \sup_Q \{fb \mid b \in B \;\&\; (\exists x \in C)(b \leq x)\} \\ &= \sup_Q \{\sup_Q f\, B_x \mid x \in C\} \\ &= \sup_Q g\, C. \end{aligned}$$

Therefore g is a continuous extension of f. Any such must satisfy (4), so g is unique. □

Chain-complete posets that satisfy (1) and (2) in the above theorem have sometimes been called "algebraic." Thus the theorem implies that P has an extension basis iff P is algebraic.

Example 3.4 A countable complete lattice need not have an extension basis. Let P be $\{\bot, \top, a_1, a_2, \cdots, b_1, b_2, \cdots\}$, ordered as shown in Fig. 1. The set B of all compact elements of P is just $\{\bot\}$. There are two ways to apply Theorem 3.3 in showing that P lacks an extension basis. First, observe that $\sup B_\top = \bot \neq \top$, contrary to (2) in Theorem 3.3. Second, observe that P-$\{\top\}$ is the set of all chain-irreducible elements of P, contrary to (3) in Theorem 3.3. The reader may also find it instructive to derive the lack of an extension basis directly from Definition 3.1. □

Example 3.5 Partial function posets have extension bases. Let X, Y be sets, and let P be the set of all partial functions mapping X into Y. Considering partial functions as subsets of $X \times Y$, we partially order P by set inclusion. Then P is a chain-complete poset, as is well known. The compact elements are those that are finite subsets of $X \times Y$. Theorem 3.3 provides an extension basis. □

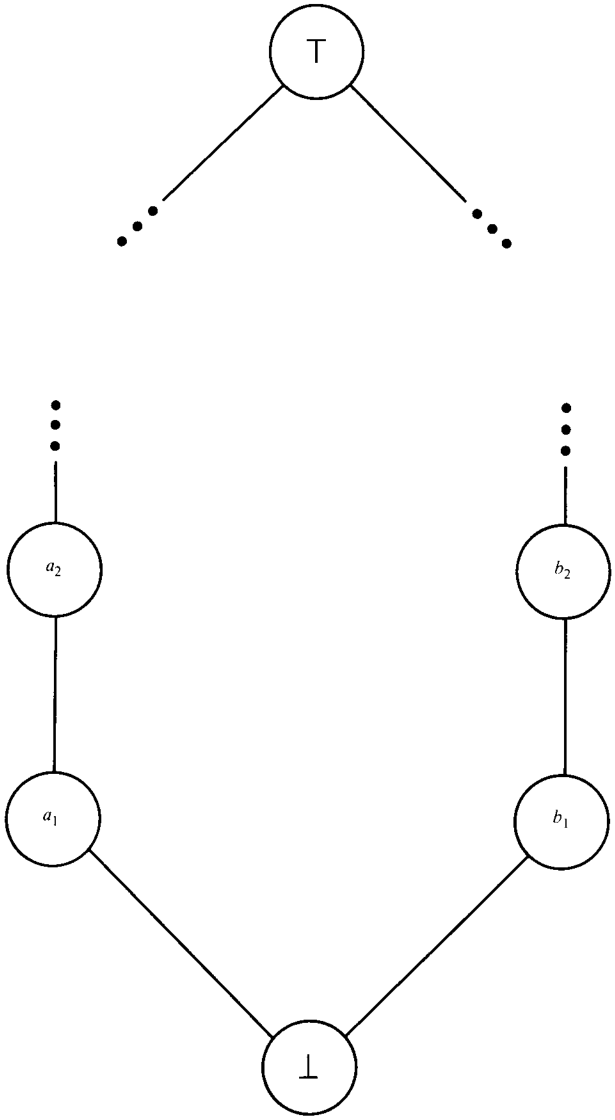

Figure 1 Countable complete lattice with no extension basis.

4. Spaces of continuous maps

If P and Q are chain-complete posets, then the set of all chain-continuous maps $f{:}P \rightarrow Q$ becomes a chain-complete poset $[P \rightarrow Q]$ under the usual ordering: $f \leq g$ in $[P \rightarrow Q]$ iff $fx \leq gx$ in Q for all x in P. We investigate conditions under which $[P \rightarrow Q]$ has an extension basis.

Lemma 4.1 Let P, Q be chain-complete posets, and let $p \in P$, $q \in Q$. Specify $f(p, q){:}P \rightarrow Q$ by

$f(p, q)x = (\textbf{if } x \geq p \textbf{ then } q \textbf{ else } \bot)$.

If p is compact, then $f(p, q)$ is chain-continuous. If q is also compact, then $f(p, q)$ is compact in $[P \rightarrow Q]$. □

Example 4.2 The existence of extension bases for P and Q does not imply the existence of extension bases for $[P \rightarrow Q]$. Let P be $\{\bot, a, b, c_1, c_2, \cdots\}$, ordered as shown in Fig. 2. For each f in $[P \rightarrow P]$, we show that

fP infinite implies f not compact. (1)

Let C be $\{c_1, c_2, \cdots\}$. For each i let f_i have

$f_i x = fx$ **if** $(fx \notin C$ or $fx \geq c_i)$;

$f_i x = c_{j+1}$ **if** $(fx = c_j$ for $j > i)$.

Then $\{f_1, f_2, \cdots\}$ is a chain whose supremum is f, but $f_i \neq f$ for all i.

Let $g{:}P \rightarrow P$ with $gx = x$, and let B_g be the set of all compact f in $[P \rightarrow P]$ such that $f \leq g$. We show that

B_g is not directed. (2)

Consider $f(a, a)$ and $f(b, b)$ from Lemma 4.1. Both are in B_g. Any upper bound f for $\{f(a, a), f(b, b)\}$ must have $fa \geq a$, $fb \geq b$, and hence $fC \subseteq C$. But $f \leq g$ then implies that fP is infinite. By (1), f cannot be compact.

Because all members of P are compact, Theorem 3.3 implies that it has an extension basis. But (2) and Theorem 3.3 imply that $[P \rightarrow P]$ does not have an extension basis. □

Are there natural conditions under which $[P \rightarrow Q]$ has an extension basis? The following property is possessed by any lattice and by many posets that are not lattices, such as the partial function poset from Example 3.5.

Definition 4.3 A poset P has *bounded joins* iff every finite subset of P with an upper bound has a supremum.

By Lemma 2.3, if P is chain-complete and has bounded joins, then every bounded subset of P has a supremum.

Lemma 4.4 Let P, Q be chain-complete posets. For any $A \subseteq [P \rightarrow Q]$ and $x \in P$, let Ax be $\{fx | f \in A\}$. Then the condition

$(\forall x \in P)$ (Ax has a supremum in Q) (1)

implies that

A has a supremum in $[P \rightarrow Q]$ (2)

and that all x in P have

$(\sup A)x = \sup(Ax)$. (3)

If Q has bounded joins, then so does $[P \rightarrow Q]$, and then (2) implies (1) also.

Proof Assume (1) and set $gx = \sup(Ax)$ for each x. Use associativity of supremums in Q and continuity of each $f \in A$ to calculate that g is continuous. It is clearly the supremum of A. Now suppose Q has bounded joins and $h = \sup A$. Then hx is an upper bound of Ax and (1) follows. □

Theorem 4.5 Let P, Q be chain-complete posets with extension bases B, C (respectively). Suppose that either P or Q has bounded joins. Then $[P \to Q]$ has an extension basis Y, where

$$Y = \{\sup A \mid A \subseteq F \ \&\ A \text{ is finite } \& \ A \text{ has a supremum in } [P \to Q]\}, \quad (1)$$

and

$$F = \{f(p, q) \mid p \in B \ \&\ q \in C\}. \quad (2)$$

Proof Let X be the set of all compact members of $[P \to Q]$. Let Y, F be as in (1), (2). Then $Y \subseteq X$ by Lemma 4.1 and the fact that a supremum of finitely many compact items is compact. We show that Y is an extension basis.

For each x in P let B_x be $\{b \in B \mid b \leq x\}$. Define C_y for each Y in Q and Y_h for each h in $[P \to Q]$ similarly. We show that

$$\sup Y_h = h \text{ for all } h \text{ in } [P \to Q]. \quad (3)$$

Of course, h is an upper bound for Y_h. By Theorem 3.3 for Q, each b in B has

$$hb = \sup C_{hb} = \sup \{f(b, c)b \mid c \in C_{hb}\}. \quad (4)$$

Any c in C_{hb} has $f(b, c)$ in Y_h, so (4) implies that any upper bound u for Y_h has $hb \leq ub$. This holds for all b in B, so Lemma 2.3 and Theorem 3.3 for P imply that $h \leq u$ for any upper bound u. This proves (3).

We show that

$$Y_h \text{ is directed for all } h \text{ in } [P \to Q]. \quad (5)$$

If Q has bounded joins, then so does $[P \to Q]$ by Lemma 4.4, and (5) follows readily from the associativity of sup. Now suppose instead that P has bounded joins. For any $g_1 = \sup A_1$ and $g_2 = \sup A_2$ in Y_h, we seek g_3 in Y_h with $g_1 \leq g_3$ and $g_2 \leq g_3$.

The set

$$M = \{b \in B \mid (\exists c \in C)(f(b, c) \in A_1 \cup A_2)\}$$

is finite, and every subset of the form $M \cap B_x$ for x in P has a supremum in P and indeed in B_x. Let

$$N = \{\sup (M \cap B_x) \mid x \in P\} \subseteq B,$$

and list the members of N as $(s_1, \cdots, s_k)$ in B^k in such a way that $s_i < s_j$ implies $i < j$. Now $(t_1, \cdots, t_k)$ in C^k is specified by induction. Recall that C_{hs_j} is directed and that $C_{hs_i} \subseteq C_{hs_j}$ whenever $s_i \leq s_j$. Given t_i for all $i < j$, it is possible to choose t_j in C_{hs_j} with $g_1 s_j \leq t_j$ and $g_2 s_j \leq t_j$ for all i such that $s_i < s_j$. The finite subset

$$A_3 = \{f(s_j, t_j) \mid 1 \leq j \leq k\}$$

of F is now shown to have a supremum g_3. For each x in P, $\sup(M \cap B_x) = s_j$ for some unique j and $A_3 x$ has a greatest element, namely t_j. Therefore $A_3 x$ has a supremum for all x and so $g_3 = \sup A_3$ exists in Y by Lemma 4.4. For each x in P, the j with $s_j = \sup(M \cap B_x)$ has $g_3 x = t_j \leq hs_j \leq hx$, so g_3 is in Y_h. Because $g_1 x = g_1 s_j$ also, $g_1 \leq g_3$. Similarly, $g_2 \leq g_3$. This proves (5).

From (3) and (5) it follows first that $Y = X$ (by Lemma 2.6) and then that Y is an extension basis (by Theorem 3.3). □

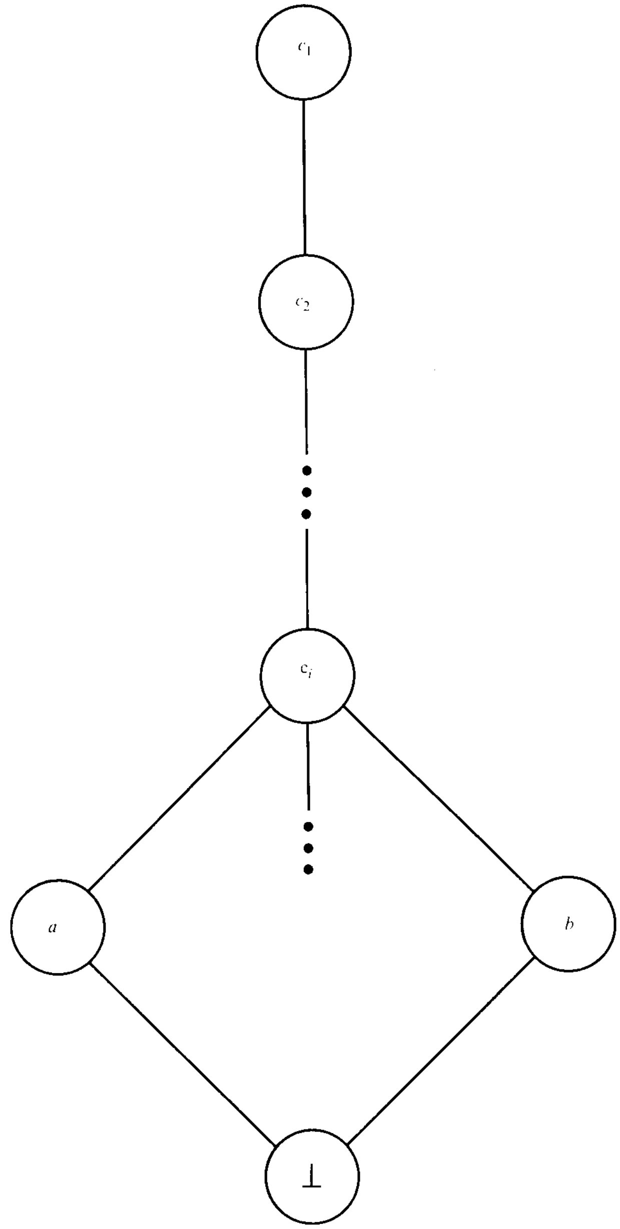

Figure 2 For each i, $a < c_i$ and $b < c_i$ and $c_{i+1} < c_i$.

The bounded joins assumption and the full force of Theorem 3.3 were only used to obtain directedness above. Under weaker conditions we can repeat the proof of (3) above to express any h in $[P \to Q]$ as the supremum of a very simple set of compact members of $[P \to Q]$.

Corollary 4.6 Let P, Q be chain-complete posets, and let B, C be any sets of compact members of P, Q (respectively). Suppose each x in P and each y in Q have

$$x = \sup B_x \;\&\; y = \sup C_y$$

for some $B_x \subseteq B$ and $C_y \subseteq C$, with B_x directed. Then any h in $[P \to Q]$ has

$$h = \sup \{f(b, c) | b \in B \;\&\; c \in C \;\&\; c \leq hb\}. \;\square$$

For use in the next section, we characterize those finite $A \subseteq F$ that have supremums in Theorem 4.5 for Q with bounded joins. Let P, Q, B, C, and F be as in Theorem 4.5.

Theorem 4.7 Suppose Q has bounded joins. For any finite $A \subseteq F$, consider the set

$$\Pi A = \{b \in B | (\exists c \in C)(f(b, c) \in A)\}. \tag{1}$$

For each $R \subseteq \Pi A$ with an upper bound in P, let

$$\Sigma(A, R) = \{c \in C | f(b, c) \in A \;\&\; b \in R\}. \tag{2}$$

Then A has a supremum in $[P \to Q]$ iff

$$\Sigma(A, R) \text{ has a supremum in } Q \text{ for all bounded } R, \tag{3}$$

and in that case

$$(\sup A)x = \sup \Sigma(A, B_x \cap \Pi A) \text{ for all } x \text{ in } P. \tag{4}$$

For any finite $A_1, A_2 \subseteq F$ with supremums in $[P \to Q]$, $\sup A_1 \leq \sup A_2$ iff each $f(b_1, c_1) \in A_1$ has

$$c_1 \leq \sup \{c_2 \in C | (\exists b_2 \leq b_1)(f(b_2, c_2) \in A_2)\}. \tag{5}$$

Proof Note that each x in P has

$$Ax = \Sigma(A, B_x \cap \Pi A), \tag{6}$$

and that each $R \subseteq \Pi A$ with an upper bound in P has $R \subseteq B_x \cap \Pi A$ for some x. Because Q has bounded joins, a supremum for $\Sigma(A, B_x \cap \Pi A)$ implies a supremum for $\Sigma(A, R)$. By (6) and Lemma 4.4, A has a supremum iff (3) holds, and in that case (4) holds.

Now suppose $\sup A_1 \leq \sup A_2$, and consider any $f(b_1, c_1) \in A_1$. Then (4) for A_2 yields

$$c_1 = f(b_1, c_1)b_1 \leq \sup \Sigma(A_2, B_{b_1} \cap \Pi A_2)$$
$$= \sup \{c_2 \in C | (\exists b_2 \leq b_1)(f(b_2, c_2) \in A_2)\}.$$

This proves (5). Now suppose (5) for each $f(b_1, c_1) \in A_1$, so that (4) for A_2 yields

$$c_1 \leq (\sup A_2)b_1 \leq (\sup A_2)x$$

whenever $b_1 \leq x$. By (4) for A_1,

$$(\sup A_1)x = \sup \Sigma(A_1, B_x \cap \Pi A_1) \leq (\sup A_2)x. \;\square$$

5. Recursively based posets

A recursive listing of a subset B of a poset P maps the nonnegative integers onto B in such a way that order related questions about members of B can be answered by algorithms that compute with the integers.

Definition 5.1 Let $\mathbf{N}$ be the set of nonnegative integers and P be a poset. A *recursive listing* of a subset B of P is a surjection $\beta:\mathbf{N} \to B$ such that, given any i, j in $\mathbf{N}$ and any finite $M \subseteq \mathbf{N}$, it is decidable whether

$$\beta i = \bot; \tag{1}$$

$$\beta_i \leq \beta j; \tag{2}$$

$$\beta M \text{ has an upper bound in } P; \tag{3}$$

$$\beta M \text{ has an upper bound in } B; \tag{4}$$

$$\beta M \text{ has a supremum in } P; \tag{5}$$

$$\sup_P \beta M \text{ is in } B; \tag{6}$$

$$\beta i = \sup_P \beta M. \tag{7}$$

Restated more formally, Definition 5.1(1) requires that there be a $\{0, 1\}$-valued recursive function f such that, for all i in $\mathbf{N}$, $fi = 1$ iff $\beta i = \bot$. For any of the usual surjections $\lambda:\mathbf{N} \to \{M \subseteq \mathbf{N} | M \text{ finite}\}$, Definition 5.1(3) requires that there be a $\{0, 1\}$-valued recursive function g such that, for all k in $\mathbf{N}$, $gk = 1$ iff $\beta\lambda k$ has an upper bound in P. The other conditions can be restated similarly.

Note that β is not required to be injective or to be in any sense "computable." Members of P need not be integers or objects represented as integers in any agreed upon way, so it is meaningless to require that β itself be "computable" in any absolute sense. For some choices of P we might wish to require computability *relative* to other maps.

Definition 5.2 A chain-complete poset P is *recursively based* iff there is an extension basis $B \subseteq P$ and a recursive listing of B.

In Example 3.5, if X and Y are countable, then P is recursively based.

Theorem 5.3. Let P, Q be recursively based chain-complete posets. Suppose that Q has bounded joins. Then $[P \to Q]$ is recursively based and has bounded joins.

Proof Let B, C be extension bases for P, Q with recursive listings β, γ. Theorem 4.5 provides an extension basis Y for $[P \to Q]$ described in (1) and (2) from Theorem 4.5. Bounded joins for Q implies bounded joins for $[P \to Q]$. We must show that Y has a recursive listing $\eta:\mathbf{N} \to Y$.

Let λ be any of the usual surjections

$$\lambda: \mathbf{N} \to \{S \subseteq \mathbf{N} \times \mathbf{N} | S \text{ finite}\},$$

so that there is a surjection $\sigma: \mathbf{N} \to \{A \subseteq F | A \text{ finite}\}$ with

$$\sigma i = \{f(\beta j, \gamma k) | (j, k) \in \lambda i\}.$$

Then there is a surjection $\eta: \mathbf{N} \to Y$ with

$\eta i =$ [**if** (σi has a supremum in $[P \to Q]$)

then sup (σi)

else $\perp_{[P \to Q]}$].

We show that η is a recursive listing.

Consider first the problems of deciding

1. Whether σi has a supremum in $[P \to Q]$;
2. Whether $\sup(\sigma i) = \perp$.

Given i, we can find λi, then $\pi i = \{j | (\exists k)((j, k) \in \lambda i)\}$. For each $M \subseteq \pi i$, we can decide whether βM has an upper bound in P and, if so, whether $\Sigma(\sigma i, \beta M)$ in Theorem 4.7(2) has a supremum in Q. By Theorem 4.7, we can decide 1). Decidability of 2) follows from

$$\sup(\sigma i) = \perp \text{ iff } (\gamma k = \perp \text{ for all } (j, k) \in \lambda i).$$

By deciding 1) and 2) we can decide whether $\eta i = \perp$, as required by Definition 5.1(1). For Definition 5.1(2), we also apply Theorem 4.7. To decide whether $\eta i_1 \leq \eta i_2$ we need only decide whether all (j_1, k_1) in λi_1 have

$$\gamma k_1 \leq \sup \{\gamma k_2 | (j_2, k_2) \in \lambda i_2 \;\&\; \beta j_2 \leq \beta j_1\},$$

and this can easily be done.

Because Q has bounded joins and because a supremum of finitely many members of Y is in Y, the other requirements of Definition 5.1 can be met by demonstrating the following claims. Given finite $M \subseteq \mathbf{N}$ and $j \in \mathbf{N}$, it is decidable

3. Whether $\bigcup_{i \in M} \sigma i$ has a supremum,

and, if so,

4. Whether $\eta j \leq \sup \left(\bigcup_{i \in M} \sigma i \right) \leq \eta j$.

The decidability of 3) follows from Theorem 4.7, as for 1). The decidability of 4) follows from Theorem 4.7, as for Definition 5.1(2). □

We have actually proved more than the bare fact that $[P \to Q]$ is recursively based. Given algorithms for deciding whether $\beta i = \perp$, whether $\gamma i \leq \gamma j$, and so on, we have shown how to construct algorithms for deciding whether $\eta i = \perp$, and so on. Given (j, k) in $\mathbf{N} \times \mathbf{N}$, we can effectively find i such that $\eta i = f(\beta j, \gamma k)$. We summarize these facts in the following corollary.

Corollary 5.4 Let P, Q be recursively based chain-complete posets. Suppose Q has bounded joins. There is an effective construction of a recursive listing and associated decision procedures for the extension basis of $[P \to Q]$ from such listings and procedures for the extension bases of P and Q. □

Even for very simple choices of P and Q such that P has bounded joins but Q lacks bounded joins, there can be no effective construction of the above kind. Before proving this, it is convenient to consider an example showing the importance of bounded joins in Lemma 4.4.

Example 5.5 Supremums in $[P \to Q]$ need not be calculable pointwise when Q lacks bounded joins. Let P be $\{\perp, a, b\}$ with $\perp < a < b$ (so that P is a lattice). Let Q be $\{\perp, a_1, a_2, a_3, c, b\}$, ordered as shown in Fig. 3. For $i = 1, 2$ let $f_i: P \to Q$ with $f_i \perp = \perp$, $f_i a = a_i$, and $f_i b = b$. Then $\{f_1, f_2\} \subseteq [P \to Q]$ with supremum g such that $g \perp = \perp$, $g\, a_i = a_3$, and $g\, b = b$. However, $\{f_1 a, f_2 a\} = \{a_1, a_2\}$ and has no supremum in Q. □

Theorem 5.6 There is a finite recursively based chain-complete poset P with bounded joins and a countable family $\{Q_k | k \in \mathbf{N}\}$ of recursively based chain-complete posets such that each $[P \to Q_k]$ is recursively based but there is no effective construction of a recursive listing and associated decision procedures for the extension basis of $[P \to Q_k]$ from such listings and procedures for P and for Q_k.

Proof Consider any enumeration of the deterministic Turing machines and their input tapes. Let $H: \mathbf{N} \times \mathbf{N} \to \{0, 1\}$ with $H(k, r) = 1$ iff the kth machine halts on the kth input after exactly r steps. Thus H is recursive whereas the function $T: \mathbf{N} \to \{0, 1\}$ with

$$Tk = 1 \text{ iff } (\exists r \in \mathbf{N})(H(k, r) = 1)$$

is not recursive, as is well known.

Let P be as in Example 5.5. For each k, let Q_k have the same elements as Q in Example 5.5, but with $\{c_r | r \in \mathbf{N}\}$ rather than just c. All r have $a_1, a_2 < c_r$. There is no order relation between a_3 and c_r or between c_r and c_s for $r \neq s$. The other order relations of Q still hold in Q_k:

$$\perp < a_1, a_2 < a_3 < b.$$

Finally,

$$c_r < b \text{ in } Q_k \text{ iff } H(k, r) = 1.$$

Letting f_1, f_2 be as in Example 5.5, we obtain

$$\{f_1, f_2\} \text{ has a supremum iff } Tk = 0. \qquad (1)$$

Because P, Q_k and $[P \to Q_k]$ consist entirely of compact elements, Theorem 3.3 implies that these spaces are extension bases for themselves. Choose a recursive listing β for P and a recursive listing γ_k for each Q_k. Using a universal Turing machine, this can be done in a

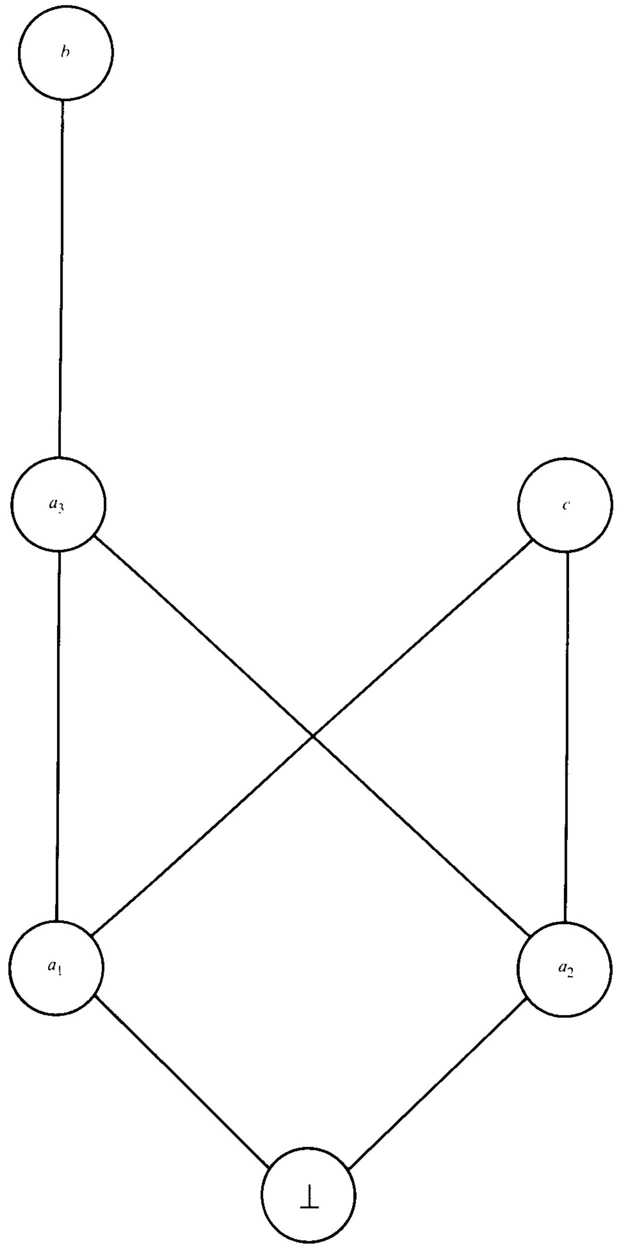

Figure 3 Poset without bounded joins.

uniform way. Not only does a recursive function g_k:N → {0, 1} have

$\gamma_k j = \bot$ in Q_k iff $g_k j = 1$,

but there is also a recursive function G: $\mathbf{N} \times \mathbf{N} \rightarrow \{0, 1\}$ with

$G(k, j) = g_k(j)$ for all k, j.

Similarly for the other conditions in Definition 5.1.

With the aid of an oracle for the halting problem, we can construct a recursive listing η_k for $P \rightarrow Q_k$. The oracle supplies Tk. As (1) illustrates, the requirements of Definition 5.1 can easily be met when this bit of information is available.

We claim the oracle is necessary: No effective construction can pass for all k from β and γ_k (with associated decision procedures) to a listing δ_k of $[P \rightarrow Q_k]$ (with associated decision procedures). Suppose otherwise. We may assume

$\beta 1 = a$ & $\beta 2 = b$;

$\gamma_k 1 = a_1$ & $\gamma_k 2 = a_2$ & $\gamma_k 4 = b$ for all k.

Given k, effectiveness permits the finding of

$r_1, r_2, s \in \mathbf{N}$

such that

$\delta_k r_1 = f(\beta 1, \gamma_k 1) = f(a, a_1)$;

$\delta_k r_2 = f(\beta 1, \gamma_k 2) = f(a, a_2)$;

$\delta_k s = f(\beta 2, \gamma_k 4) = f(b, b)$.

We can then find $i_1, i_2 \in \mathbf{N}$ such that

$\delta_k i_1 = \sup \{\delta_k r_1, \delta_k s\} = f_1$;

$\delta_k i_2 = \sup \{\delta_k r_2, \delta_k s\} = f_2$.

Now define T':$\mathbf{N} \rightarrow \{0, 1\}$ by

$T'k = 1$ iff $\{\delta_k i_1, \delta_k i_2\}$ has no supremum,

so that T' is recursive. But $T = T'$ by (1) and T is not recursive. □

In order to compare Theorem 5.3 with the results in [14, 15] we must introduce a property stronger than possession of bounded joins.

Definition 5.7 A subset A of a poset P is *pairwise compatible* iff every $\{x, y\} \subseteq A$ has an upper bound in P. A poset P is *coherent* iff every pairwise compatible $A \subseteq P$ has a supremum.

In particular, a coherent poset is chain-complete. Using Corollary 2.9 to bridge the gap between ω-completeness and completeness, it is not hard to show that P is a coherent recursively based poset iff P is a "cpo with a recursive basis" [14, Sec. II.2] iff P is a "domain of calculation" [15, Chap. III]. Thus the following corollary is equivalent to the theorem in [14, Sec. II.2] and to Lemma 2 in [15, Chap. III].

Corollary 5.8 Let P and Q be coherent recursively based posets. Then $[P \rightarrow Q]$ is a coherent recursively based poset. □

Coherence is a useful property. The partial function poset of Example 3.5 is coherent, and this fact has been used countless times. To construct a total function $F: X \rightarrow Y$, it suffices to construct a family $\mathcal{F}$ of partial functions $f: X_f \rightarrow Y$ such that

$$X = \bigcup_{f \in \mathcal{F}} X_f$$

and such that any f, g in $\mathcal{F}$ have $fx = gx$ whenever $x \in X_f \cap X_g$. This last condition is pairwise compatibility of $\mathcal{F}$. The function F is the supremum of $\mathcal{F}$. Coherence is therefore interesting in its own right. As we have shown, it is not necessary in studying recursive listings of extension bases. It is not even helpful, except as a sufficient condition for the possession of bounded joins.

A very slight simplification can be achieved by assuming that Q is a lattice: we need not bother deciding whether subsets of Q have upper bounds. A chain-complete poset Q with bounded joins can easily be made into a complete lattice Q^+ by adding a new top element [2, Sec. 4]. To do this prematurely may cause embarassment. With one new element we can make $[P \rightarrow Q]$ into a lattice $[P \rightarrow Q]^+$. The lattice $[P \rightarrow Q^+]$ is cluttered with a great many more new elements that map some members of P into Q and others into $Q^+ - Q$. Ockham's razor would have us postpone the addition of an ad hoc top until there is a definite need for it.

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The authors are located at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598.