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Abstract: A comparative study is made of algorithms for a general multidimensional problem involving the packing of k-part objects in k compartments in a large supply of bins. The goal is to pack the objects using a minimum number of bins. The properties and limitations of the algorithms are discussed, including k-dimensional analogs of some popular one-dimensional algorithms. An application of the algorithms is the design of computer networks.

Introduction

Consider a collection of k-part objects, each part of every object having size between 0 and 1, and a large supply of bins, each divided into k compartments and each compartment with capacity 1. The bin packing problem is to pack the objects into as few bins as possible with each part of every object going into the correct compartment in a bin. The one-dimensional version of this problem (i.e., $k = 1$) has been studied thoroughly [1-5]; it has applications in operations research [6-9], computer operating system design, and memory allocation [1, 2]. Little is known about the general k-dimensional problem, which has applications to computer network design [10]. To formalize our problem, we begin by introducing some notation.

Definition a) For $k \geq 1$, let $\theta^k = \{(a_1, \cdots, a_k) | \text{ all } i, 0 \leq a_i \leq 1\}$; b) Let $\mathbf{n} = \{1, \cdots, n\}$; and c) Given a function $v: \mathbf{n} \rightarrow \theta^k$, and a partition $\mathscr{L} = \{S_1, \cdots, S_m\}$ of $\mathbf{n}$, we say that $\mathscr{L}$ is *v-admissible* if for all $j \in \mathbf{m}$, $\Sigma_{i \in S_j} v(i) \in \theta^k$.

Remark Observe that admissible partitions correspond to bin packings, where the k-part objects are represented by elements of θ^k. Thus, any bin packing algorithm yields some admissible partition, and we infer properties of the algorithm from properties of the corresponding partition. In this paper we are interested in analyzing all "reasonable" algorithms. One natural criterion for reasonable algorithms is that they produce "irreducible" partitions, a concept which we define now.

Definition Let $v: \mathbf{n} \rightarrow \theta^k$ be a function and $\mathscr{L} = \{S_1, \cdots, S_m\}$ be a partition of $\mathbf{n}$. a) We say that $\mathscr{L}$ is *v-irreducible* if $\mathscr{L}$ is v-admissible, and for all $i, j \in \mathbf{m}$, $i \neq j$ implies

$$\sum_{\lambda \in S_i \cup S_j} v(\lambda) \not\in \theta^k.$$

b) We let $\mathscr{C}(v)$ be the set of all v-irreducible partitions.
c) We let $\text{opt}(v) = \min \{|\mathscr{L}| | \mathscr{L} \in \mathscr{C}(v)\}$.

Clearly, $\text{opt}(v)$ gives the smallest number of bins into which the list of vectors $v(1), \cdots, v(n)$ can be packed, since any optimal partition must be irreducible.

For every fixed integer $k \geq 1$, it is not hard to show that the following formulation of the bin packing problem is NP-complete in the sense of Cook [6] and Karp [14].

Let the input be a function $v: \mathbf{n} \rightarrow \theta^k$ and a positive integer l, and then try to determine if there exists a v-admissible partition $\mathscr{L}$ of size l.

Hence, another natural criterion for reasonable bin packing algorithms is that each has its running time bounded by some polynomial of the input length. In this paper, by a reasonable bin packing algorithm, we mean that 1) it produces v-irreducible partitions and 2) it possesses polynomially bounded running times. We will show that constraint 1) implies for all $\mathscr{L} \in \mathscr{C}(v)$ $|\mathscr{L}|/\text{opt}(v) \leq k + 1$. Furthermore, for each $k \geq 1$ and $\delta > 0$, we can find an n, v, and $\mathscr{L} \in \mathscr{C}(v)$ such that $|\mathscr{L}|/\text{opt}(v) \geq (k + 1 - \delta)$; i.e., $k + 1$ is a sharp upper bound (independent of n) on the ratio $|\mathscr{L}|/\text{opt}(v)$. It thus follows that any reasonable algorithm will do no worse than $(k + 1)$ times the optimal packing in terms of the number of bins used. On the other hand, since whether all NP-complete problems have polynomial-time-bounded solutions is still an open question at the present time, we are not able to show any lower bound on the ratio $|\mathscr{L}|/\text{opt(v)}$. However, we show that if NP-complete problems do not have polynomial-time-bounded solutions, a most likely result, 2) implies that all reasonable heuristic algorithms will produce in the worst case at least 50 percent more than the optimal number of bins.

For the one-dimensional case, several algorithms have been noted, namely, the first fit algorithm, the best fit algorithm, the first fit decreasing algorithm and the best fit decreasing algorithm [1-5]. For convenience, we state the k-dimensional versions of these algorithms. We have a function $v:\mathbf{n} \rightarrow \theta^k$, and an infinite collection of bins $S_1, S_2, \cdots$, which at stage 0 are all empty. At stage i, we place i in one of the bins according to one of the four rules subsequently described. (After stage n, the non-empty bins yield a partition of $\mathbf{n}$. We use S_i^j to indicate the set of numbers in S_i at stage j.)

- *First Fit Algorithm (FF)*

At stage j, for $j \geq 1$, place j in S_λ, where λ is the least integer such that $v(j) + \Sigma_{t \in S_\lambda^{j-1}} v(t) \in \theta^k$.

- *Best Fit Algorithm (BF)*

At stage j, for $j \geq 1$, place j in S_λ, where λ is the least integer such that

1. $v(j) + \Sigma_{t \in S_\lambda^{j-1}} v(t) \in \theta^k$, and
2. for all i, $v(j) + \Sigma_{t \in S_i^{j-1}} v(t) \in \theta^k$ implies that the sum of the k components of $\Sigma_{t \in S_i^{j-1}} v(t)$ is not greater than the sum of the k components of $\Sigma_{t \in S_\lambda^{j-1}} v(t)$.

- *First Fit Decreasing Algorithms (FFD_1, FFD_2, FFD_3)*

We specify a linear quasi-order, $\preceq$, on θ^k, and permute the domain of v so that $i \preceq j$ implies that $v(j) \preceq v(i)$. Then we apply the FF Algorithm. The following are a few quasi-orders that we consider in this paper. Let $a, b \in \theta^k$.

Lexicographical FFD (FFD_1)
$a \preceq b$ iff $a = b$ or the first nonzero component of $b - a$ is positive.
Maximum Component FFD (FFD_2)
$a \preceq b$ iff the maximum component in b is not less than the maximum component of a.
Maximum Sum FFD (FFD_3)
$a \preceq b$ iff the sum of the components of b is not less than the sum of components of a.

- *Best Fit Decreasing Algorithms (BFD_1, BFD_2, BFD_3)*

We proceed exactly as with the First Fit Decreasing Algorithms, but we apply the BF rather than the FF Algorithm whenever appropriate.

We show below that for each k and for every $\delta > 0$, there exists a function $v:\mathbf{n} \rightarrow \theta^k$, such that all the algorithms above yield the same partition, $\Delta(v)$, and that $|\Delta(v)|/\text{opt}(v) \geq (k - \delta)$. Thus, these algorithms do little, if at all, better than any reasonable algorithm, in the worst case.

Upper bound

We use the following conventions throughout this paper.

1. Given two sets, S and T, $S - T$ denotes the set of all elements in S and not in T.
2. If S is the empty set, we arbitrarily set $\min S = \max S = 0$. Otherwise, if S is a non-empty set of real numbers, then $\min S$ ($\max S$) denotes the least (greatest) element in S.
3. If $v:\mathbf{n} \rightarrow \theta^k$, then for $i \in \mathbf{n}, j \in \mathbf{k}$, $v_j(i)$ denotes the jth component of $v(i)$.
4. We use ε^k (or simply ε, if k is obvious) to denote the vector in θ^k with all components equal to ε.
5. We use I_j^k (or simply I_j, if k is obvious) to denote the vector in θ^k with all components zero, except for the jth component, which is 1.
6. For all real numbers r, $\lceil r \rceil$ denotes the least integer i such that $i \geq r$, and $\lfloor r \rfloor$ denotes the greatest integer i such that $i \leq r$.

Lemma 1 [13] For every function $f:\mathbf{m} \rightarrow \{r | r \geq 0\}$, there exists a function $g:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}$ such that, for all $i, j \in \mathbf{m}$. The statement $f(i) + f(j) \geq 1$ implies that $g(i) + g(j) \geq 1$, and $\Sigma_{i=1}^m g(i) \leq \Sigma_{i=1}^m f(i)$.

Proof Let

$$U_f = \{f(i) | i \in \mathbf{m} \text{ and } f(i) \not\in \{0, \tfrac{1}{2}, 1\}\},$$

$$L_f = \{i | f(i) \in U_f \text{ and } f(i) > \tfrac{1}{2}\},$$

$$S_f = \{i | f(i) \in U_f \text{ and } f(i) < \tfrac{1}{2}\},$$

$$V_f = \{i | i \in L_f \text{ and } f(i) + \min\{f(j) | j \in S_f\} \geq 1\},$$

$$\alpha_f = \begin{cases} \max\{f(i) | i \in (L_f - V_f)\} & \text{if } L_f \neq V_f, \\ \tfrac{1}{2} & \text{otherwise,} \end{cases}$$

and

$$W_f = \{i | i \in S_f \text{ and } \alpha_f + f(i) < 1\}.$$

The proof proceeds by induction on the size of U_f. If $|U_f| = 0$, by setting $g = f$, the lemma is trivially true. Let us suppose that the lemma is true whenever $|U_f| \leq p$, and assume then $|U_f| = p + 1$. We now consider three cases.

Case 1 $|W_f| = 0$. We first observe that, by definition of W_f, $|W_f| = 0$ implies $S_f = \emptyset$. Consider the function $g:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}$ such that, for all $i \in \mathbf{m}$,

$$g(i) = \begin{cases} f(i) & \text{if } f(i) \not\in U_f, \\ \tfrac{1}{2} & \text{if } f(i) \in U_f \text{ and } f(i) < 1, \text{ and} \\ 1 & \text{if } f(i) \in U_f \text{ and } f(i) > 1; \end{cases}$$

g certainly has the required property. The lemma follows.

Case 2 $|V_f| \leq |W_f| \neq 0$. Consider the function $f':\mathbf{m} \rightarrow \{r | r \geq 0\}$ such that, for all $i \in \mathbf{m}$,

$$f'(i) = \begin{cases} f(i) & \text{if } i \not\in V_f \cup W_f, \\ 1 & \text{if } i \in V_f, \text{ and} \\ 0 & \text{if } i \in W_f. \end{cases}$$

Notice that the size of the set $U_{f'} = \{f'(i) | i \in \mathbf{m}$ and $f'(i) \not\subseteq \{0, \frac{1}{2}, 1\}\}$ is no greater than p. By induction hypothesis, there exists a function $g:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}$ such that, for all $i, j \in \mathbf{m}$, $f'(i) + f'(j) \geq 1$ implies $g(i) + g(j) \geq 1$ and $\Sigma_{i=1}^{m} g(i) \leq \Sigma_{i=1}^{m} f'(i)$. By the construction of f', for all $i, j \in \mathbf{m}$, $f(i) + f(j) \geq 1$ implies $f'(i) + f'(j) \geq 1$ and $\Sigma_{i=1}^{m} f'(i) \leq \Sigma_{i=1}^{m} f(i)$. The lemma follows.

Case 3 $|V_f| > |W_f| \neq 0$. Consider the function $f'':\mathbf{m} \rightarrow \{r | r \geq 0\}$ such that, for all $i \in \mathbf{m}$,

$$f''(i) = \begin{cases} f(i) & \text{if } i \notin V_f \cup W_f, \\ \alpha_f & \text{if } i \in V_f, \text{ and} \\ 1 - \alpha_f & \text{if } i \in W_f. \end{cases}$$

Notice that the size of the set $U_{f''} = \{f''(i) \mid i \in \mathbf{m}$ and $f''(i) \not\subseteq \{0, \frac{1}{2}, 1\}\}$ is no greater than p. By induction hypothesis, there exists a function $g:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}$ such that, for all $i, j \in \mathbf{m}$, $f''(i) + f''(j) \geq 1$ implies $g(i) + g(j) \geq 1$ and $\Sigma_{i=1}^{m} g(i) \leq \Sigma_{i=1}^{m} f''(i)$. By the construction of f'', for all $i, j \in \mathbf{m}$, $f(i) + f(j) \geq 1$ implies $f''(i) + f''(j) \geq 1$ and $\Sigma_{i=1}^{m} f''(i) \leq \Sigma_{i=1}^{m} f(i)$. The lemma follows. □

Definition a) A function $f:\mathbf{m} \rightarrow \theta^k$ is called *final*, if for all $i, j \in \mathbf{m}$ there exists $p \in \mathbf{k}$ such that $f_p(i) + f_p(j) \geq 1$. b) For $m, k \geq 1$, we use $\mathscr{F}(m, k)$ to denote the set of all final functions $f:\mathbf{m} \rightarrow \theta^k$. c) For $m, k \geq 1$, we use $\rho(m, k)$ to denote $\min\{\max\{\Sigma_{i=1}^{m} f_j(i) | j \in \mathbf{k}\} | f \in \mathscr{F}(m, k)\}$.

Remark Given a function $v:\mathbf{n} \rightarrow \theta^k$ and a v-irreducible partition, $\mathscr{L} = \{S_1, \cdots, S_m\}$, the function $f:\mathbf{m} \rightarrow \theta^k$ given by $f(i) = \Sigma_{\lambda \in S_i} v(\lambda)$ is final. Observe that $m \geq \mathrm{opt}(v) \geq \max\{\Sigma_{i=1}^{n} v_j(i) \mid j \in \mathbf{k}\} = \max\{\Sigma_{i=1}^{m} f_j(i) \mid j \in \mathbf{k}\} \geq \rho(m, k)$. Hence, $|\mathscr{L}|/\mathrm{opt}(v) \leq |\mathscr{L}|/\rho(m, k) \leq m/\rho(m, k)$. We will calculate the exact value of $\rho(m, k)$ and use it to prove that $|\mathscr{L}|/\mathrm{opt}(v) \leq (k + 1)$ for all $\mathscr{L} \in \mathscr{C}(v)$.

Lemma 2 For every final function $v:\mathbf{m} \rightarrow \theta^k$, there exists a final function $u:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}^k$ such that a) for all $j \in \mathbf{k}$, $\Sigma_{i=1}^{m} u_j(i) \leq \Sigma_{i=1}^{m} v_j(i)$; and b) either the size of the set $S_u = \{i | i \in \mathbf{m}$ and $u_j(i) = 1$ for some $j \in \mathbf{k}\}$ is not less than $m - 1$, or there exists a $j_0 \in \mathbf{k}$ such that, for all $i \in \mathbf{m} - S_u$, $u_{j_0}(i) = \frac{1}{2}$.

Proof For every final function $v:\mathbf{m} \rightarrow \theta^k$, by Lemma 1, there exists a final function $v':\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}^k$ such that $\Sigma_{i=1}^{m} v'_j(i) \leq \Sigma_{i=1}^{m} v_j(i)$ for all $j \in \mathbf{k}$. Let $S_{v'} = \{i | i \in \mathbf{m}$ and $v'_j(i) = 1$ for some $j \in \mathbf{k}\}$. The proof of this lemma proceeds by induction on the size of $\mathbf{m} - S_{v'}$. If $|\mathbf{m} - S_{v'}| \leq 1$, then by setting $u = v'$, the lemma is trivially true. Suppose the lemma is true whenever $|\mathbf{m} - S_{v'}| \leq l$. Assume then $|\mathbf{m} - S_{v'}| = l + 1$. If there exists a $j_0 \in \mathbf{k}$ such that, for all $i \in \mathbf{m} - S_{v'}$, $v'_{j_0}(i) = \frac{1}{2}$, then, by setting $u = v'$, the lemma is trivially true. Otherwise, consider the following two cases:

Case 1 there exists $i_1, i_2 \in \mathbf{m} - S_{v'}$, $j_1, j_2 \in \mathbf{k}$ with $i_1 \neq i_2$ and $j_1 \neq j_2$ such that $v'_{j_1}(i_1) = v'_{j_1}(i_2) = \frac{1}{2}$ and $v'_{j_2}(i_1) = v'_{j_2}(i_2) = \frac{1}{2}$. Consider the function $w:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}^k$ such that, for all $i \in \mathbf{m}$,

$$w(i) = \begin{cases} v'(i) & \text{if } i \notin \{i_1, i_2\}, \\ I_{j_1}^k & \text{if } i = i_1, \text{ and} \\ I_{j_2}^k & \text{if } i = i_2. \end{cases}$$

Notice that the size of the set $S_w = \{i | w_j(i) = 1$ for some $i \in \mathbf{m}$ and $j \in \mathbf{k}\}$ is greater than the size of $S_{v'}$. Hence, $|\mathbf{m} - S_w| \leq l$, and we apply the induction hypothesis and the fact that $\Sigma_{i=1}^{m} w_j(i) \leq \Sigma_{i=1}^{m} v'_j(i)$ for all $j \in \mathbf{k}$.

Case 2 For any pair $i_1, i_2 \in \mathbf{m} - S_{v'}$, with $i_1 \neq i_2$, there exists a $j_1 \in \mathbf{k}$ such that $v'_{j_1}(i_1) = v'_{j_1}(i_2) = \frac{1}{2}$ and, for all $j \in \mathbf{k}$ and $j \neq j_1$, $v'_j(i_1) + v'_j(i_2) \leq \frac{1}{2}$.

In this case, there must exist an $i_3 \in \mathbf{m} - S_{v'}$ and $j_2, j_3 \in \mathbf{k}$ with $j_1 \neq j_2 \neq j_3 \neq j_1$, such that $v'_{j_1}(i_3) = v'_{j_2}(i_2) = v'_{j_3}(i_1) = 0$ and $v'_{j_2}(i_1) = v'_{j_2}(i_3) = v'_{j_3}(i_2) = v'_{j_3}(i_3) = \frac{1}{2}$, since v' is final. Consider the function $w:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}^k$ such that, for all $i \in \mathbf{m}$,

$$w(i) = \begin{cases} v'(i) & \text{if } i \notin \{i_1, i_2, i_3\}, \\ I_{j_1}^k & \text{if } i = i_1, \\ I_{j_2}^k & \text{if } i = i_2, \text{ and} \\ I_{j_3}^k & \text{if } i = i_3. \end{cases}$$

By an argument similar to that used in Case 1, the lemma follows from the induction hypothesis and the fact that $\Sigma_{i=1}^{m} w_j(i) \leq \Sigma_{i=1}^{m} v'_j(i)$, for all $j \in \mathbf{k}$. □

Theorem 1 For all $m \geq 2$ and $k \geq 1$, $\rho(m, k) = \lfloor m/(k + 1) \rfloor + l(m, k)$, where

$$l(m, k) = \begin{cases} 0 & \text{if } m \equiv 0 \bmod (k + 1), \\ \frac{1}{2} & \text{if } m \equiv 1 \bmod (k + 1), \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Proof By Lemma 2, it follows that there exists a final function $u:\mathbf{m} \rightarrow \{0, \frac{1}{2}, 1\}^k$ such that a) $\max\{\Sigma_{i=1}^{m} u_j(i) | j \in \mathbf{k}\} = \rho(m, k)$, and b) either the size of the set $S_u = \{i | u_j(i) = 1$ for some $i \in \mathbf{m}$ and $j \in \mathbf{k}\}$ is no less than $m - 1$ or there exists a $j_0 \in \mathbf{k}$ such that, for all $i \in \mathbf{m} - S_u$, $u_{j_0}(i) = \frac{1}{2}$. Let $m = \alpha(k + 1) + \beta$ where α, β are integers such that $\alpha \geq 0$ and $k \geq \beta \geq 0$. Notice that, since $m \geq 2$, we have $\alpha + \beta \geq 1$. Furthermore, $\alpha + \beta = 1$ implies $\beta = 0$. Let us first prove that $\rho(m, k) \geq \lfloor m/(k + 1) \rfloor + l(m, k)$. Consider the following two cases.

Case 1 $|S_u| \geq \mathrm{m} - 1$. In this case, we have

$$\rho(m, k) = \max\left\{\sum_{i=1}^{m} u_j(i) | j \in \mathbf{k}\right\}$$

$$\geq \lceil (m - 1)/k \rceil \qquad (\text{since } \sum_i \sum_j u_j(i) \geq m - 1)$$

$$=\lceil[\alpha(k+1)+\beta-1]/k\rceil$$
$$=\lceil \alpha+[(\alpha+\beta-1)/k]\rceil$$
$$\ge\begin{cases}\alpha & \text{if } \beta=0\text{, and}\\ \alpha+1 & \text{otherwise.}\end{cases}$$

On the other hand,

$$\lfloor m/(k+1)\rfloor+l(m,k)=\alpha+l(m,k)$$
$$=\begin{cases}\alpha & \text{if } \beta=0,\\ \alpha+\frac{1}{2} & \text{if } \beta=1\text{, and}\\ \alpha+1 & \text{otherwise.}\end{cases}$$

Obviously, $\rho(m,k)\ge\lfloor m/(k+1)\rfloor+l(m,k)$ in this case.

Case 2 $|S_u|\le m-2$ and there exists $j_0\in\mathbf{k}$ such that, for all $i\in\mathbf{m}-S_u$, $u_{j_0}(i)=\frac{1}{2}$.

Consider the set $T=\{i|u_{j_0}(i)>0\}$. If $|T|\ge 2\alpha+2l(m,k)$, then

$$\rho(m,k)=\max\left\{\sum_{i=1}^{m}u_j(i)|j\in\mathbf{k}\right\}$$
$$\ge\sum_{i=1}^{m}u_{j_0}(i)$$
$$\ge\tfrac{1}{2}\times|T|$$
$$\ge\alpha+l(m,k)$$
$$=\lfloor m/(k+1)\rfloor+l(m,k).$$

If, however, $|T|\le 2\alpha+2l(m,k)-1$, then $k\ne 1$. Otherwise we have $|S_u|\le m-2$ and $|T|\le 2\alpha+2l(m,k)-1=m-1$. The sizes of S_u and T imply that u is not final. Notice that

$$\rho(m,k)=\max\left\{\sum_{i=1}^{m}u_j(i)|j\in\mathbf{k}\right\}$$
$$\ge\max\left\{\sum_{i=1}^{m}u_j(i)|j\in\mathbf{k}-\{j_0\}\right\}$$
$$\ge\lceil(m-|T|)/(k-1)\rceil$$
$$\left(\text{since }\sum_{i=1}^{m}\sum_{j=1,j\ne j_0}^{k}u_j(i)\ge m-|T|\right)$$
$$\ge\lceil(\alpha(k-1)+\beta-2l(m,k)+1)/(k-1)\rceil$$
$$=\alpha+1$$
$$\ge\lfloor m/(k+1)\rfloor+l(m,k).$$

Hence, in all cases $\rho(m,k)\ge\lfloor m/(k+1)\rfloor+l(m,k)$. We next show that $\rho(m,k)\le\lfloor m/(k+1)\rfloor+l(m,k)$, and thus complete the proof of the theorem. For all $m\ge 2$ and $k\ge 1$, consider the function $\tau:\mathbf{m}\to\theta^k$ such that

$$\tau(i)=\begin{cases}\frac{1}{2}\times I_1 & \text{if } i\equiv 1 \bmod (k+1) \text{ or } 2 \bmod (k+1),\\ I_k & \text{if } i\equiv 0 \bmod (k+1)\text{, and}\\ I_{\beta-1} & \text{if } i\equiv\beta \bmod (k+1) \text{ and } \beta\ge 3.\end{cases}$$

Obviously, τ is final and $\Sigma_{i=1}^{m}\tau_1(i)=\max\{\Sigma_{i=1}^{m}\tau_j(i)|j\in\mathbf{k}\}$. Notice that $\Sigma_{i=1}^{m}\tau_1(i)=\lfloor m/(k+1)\rfloor+l(m,k)$. The result now follows from the definition of $\rho(m,k)$. □

Theorem 2 If $v:\mathbf{n}\to\theta^k(k\ge 1)$ and $\mathscr{L}\in\mathscr{C}(v)$, then $|\mathscr{L}|/\text{opt}(v)\le(k+1)$.

Proof We first observe that if $\text{opt}(v)=1$, $|\mathscr{L}|=1$ for all $\mathscr{L}\in\mathscr{C}(v)$ [in fact $\mathscr{C}(v)=\{\{\mathbf{n}\}\}$]. Clearly, the theorem is true in this case. Hence, we may assume $\text{opt}(v)\ge 2$. From the remark preceding Lemma 2, it follows that $|\mathscr{L}|/\text{opt}(v)\le(\max\{m/\lceil\rho(m,k)\rceil|m\ge 2,k\ge 1\})$. However, for all $m\ge 2$, and $k\ge 1$, we have $m/\lceil\rho(m,k)\rceil=m/\lceil\lfloor m/(k+1)\rfloor+l(m,k)\rceil=m/\lceil m/(k+1)\rceil\le m/[m/(k+1)]=k+1$. □

We next show that the upper bound derived in Theorem 2 is sharp.

Theorem 3 For every integer $k\ge 1$ and $\delta>0$, there exists an integer n and a function $v:\mathbf{n}\to\theta^k$, such that for some $\mathscr{L}\in\mathscr{C}(v)$, $|\mathscr{L}|/\text{opt}(v)\ge(k+1-\delta)$.

Proof Let L be a positive integer such that $L\ge[(k+1)/\delta]-1$. Let $\varepsilon=1/[(k+1)L]$ and $n=2(k+1)L$. Let $v:\mathbf{n}\to\theta^k$ be given by

$$v(i)=\begin{cases}\frac{1}{2}(1-\varepsilon)I_1 & \text{if } i\le 2L,\\ (1-\varepsilon)I_j & \text{if } jL+1\le i\le(j+1)L\\ & \qquad (\text{for } 2\le j\le k)\text{, and}\\ \varepsilon & \text{if } (k+1)L+1\le i.\end{cases}$$

Let $\mathscr{L}=\{S_1,\cdots,S_{(k+1)L}\}$, where $S_i=\{i,(k+1)L+i\}$ for all $i=1,2,\cdots,(k+1)L$. Clearly, $\mathscr{L}\in\mathscr{C}(v)$. Let $\mathscr{L}'=\{S_1',\cdots,S_{L+1}'\}$, where

$$S_i'=\begin{cases}\{2i-1,2i\}\cup\{jL+i|2\le j\le k\}, & \text{if } 1\le i\le L,\\ \{t|(k+1)L+1\le t\le 2(k+1)L\}, & \text{if } i=L+1.\end{cases}$$

Clearly, $\mathscr{L}'\in\mathscr{C}(v)$, and $|\mathscr{L}|/\text{opt}(v)\ge|\mathscr{L}|/|\tilde{\mathscr{L}}'|\ge(k+1)L/(L+1)\ge k+1-\delta$. □

Lower bound

Let $\mathscr{L}(\mathscr{A},v)$ denote the partition of $\mathbf{n}$ produced by a bin packing algorithm $\mathscr{A}$ with respect to a given function $v:\mathbf{n}\to\theta^k$. If $\mathscr{A}$ is reasonable, then $\mathscr{L}(\mathscr{A},v)$ is necessarily v-irreducible. Theorem 2 then indicates that, for any given v, the ratio $|\mathscr{L}(\mathscr{A},v)|/\text{opt}(v)$ is bounded above $k+1$, a value independent of $\mathbf{n}$. However, unless NP = P, no reasonable algorithm will do very well, as indicated by the following theorem.

Theorem 4 If NP $\ne$ P, then, for any reasonable bin packing algorithm $\mathscr{A}$, there exists a function $v:\mathbf{n}\to\theta^k$ such that $|\mathscr{L}(\mathscr{A},v)|/\text{opt}(v)\ge 3/2$.

Proof We shall show that if there exists an algorithm $\mathscr{A}$, such that $\mathscr{L}(\mathscr{A}, v)/\text{opt}(v) < 3/2$ for all $v:\mathbf{n} \rightarrow \theta^k$, then we can effectively design a specific polynomial-time-bounded algorithm to solve one NP-complete problem. The specific NP-complete problem we pick is the so-called partition problem which can be stated as follows:

Let the input be a set of n nonnegative integers $\mathscr{S} = \{x_1, x_2, \cdots, x_n\}$, and try to determine whether there exists a partition of $\mathscr{S}$ into subsets U and V such that $\Sigma_{x_j \in U} x_j = \Sigma_{x_k \in V} x_k$.

To determine whether a given input has such a property, consider the following polynomial-time-bounded algorithm:

Step 1 Check to see if any x_i is greater than $\frac{1}{2}\Sigma_{j=1}^{n} x_j$, for $i = 1, 2, \cdots, n$. If the answer is positive, $\mathscr{S}$ does not have the property. Otherwise, go to step 2.

Step 2 Evaluate the function $v:\mathbf{n} \rightarrow \theta^1$, given by

$$v(i) = \frac{2x_i}{\sum_{j=1}^{n} x_j}, \qquad \text{for all } i \in \mathbf{n}.$$

Step 3 Compute $\mathscr{L}(\mathscr{A}, v)$. If $|\mathscr{L}(\mathscr{A}, v)| = 2$, then $\mathscr{S}$ has the property. Otherwise $\mathscr{S}$ does not have the property.

Notice that $\Sigma_{i=1}^{n} v(i) = 2$ and $\mathscr{L}(\mathscr{A}, v)$ is v-irreducible, implying that $|\mathscr{L}(\mathscr{A}, v)|$ is either 2 or 3. Since $|\mathscr{L}(\mathscr{A}, v)|/\text{opt}(v) < 3/2$, so $\text{opt}(v) = 2$ if and only if $|\mathscr{L}(\mathscr{A}, v)| = 2$. □

Remark The lower bound given here is a very loose one. In fact, we never take the factor k into consideration. The improvement on this lower bound is highly likely.

Lower bounds for specific algorithms

It is worthwhile to examine the performance of the k-dimensional analogs of some popular one-dimensional bin packing algorithms (FF, BF, FFD_1, FFD_2, FFD_3, BFD_1, BFD_2, and BFD_3). We summarize the results in the following theorem.

Theorem 5 For any integer $k \geq 1$ and real number $\delta > 0$, there exists an integer n and a function $v:\mathbf{n} \rightarrow \theta^k$ such that all the FF, BF, FFD_1, FFD_2, FFD_3, BFD_1, BFD_2 and BFD_3 algorithms will yield the same partition $\Delta \in \mathscr{C}(v)$, such that $|\Delta|/\text{opt}(v) \geq (k - \delta)$.

Proof For any integer $k \geq 1$ and real number $\delta > 0$, the theorem is trivially true if $\delta \geq k$ or $k = 1$. If $k > \delta$ and $k > 1$, let the integers $M, N_1, N_2, \cdots, N_k$ be chosen in such a way that $(k/\delta) - 1 \leq M < N_1 < N_2 < \cdots < N_k$, and let $\varepsilon \leq 1/[N_k(N_k - 1)(k - 1)]$. Furthermore, set $n = \Sigma_{\lambda=1}^{k} MN_\lambda$ and let $A_j \in \theta^k$ be the vector whose jth component is equal to $1/N_j$, while all the other components are equal to ε. Now define $v:\mathbf{n} \rightarrow \theta^k$ such that, for all $i \in \mathbf{n}$,

$$v(i) = \begin{cases} A_1 & \text{if } 1 \leq i \leq MN_1, \\ A_2 & \text{if } 1 + MN_1 \leq i \leq MN_1 + MN_2, \\ \vdots & \\ A_j & \text{if } 1 + \sum_{\lambda=1}^{j-1} MN_\lambda \leq i \leq \sum_{\lambda=1}^{j} MN_\lambda, \text{ and} \\ \vdots & \\ A_k & \text{if } 1 + \sum_{\lambda=1}^{k-1} MN_\lambda \leq i \leq \sum_{\lambda=1}^{k} MN_\lambda. \end{cases}$$

It is easy to see that all the FF, BF, FFD_1, FFD_2, FFD_3, BFD_1, BFD_2 and BFD_3 algorithms will yield the same partition, $\Delta = \{S_1, S_2, \cdots, S_{kM}\} \in \mathscr{C}(v)$, where for all j, $1 \leq j \leq kM$, if $j = \alpha M + \beta$ with $0 \leq \alpha \leq k - 1$ and $0 \leq \beta \leq M$, then $S_j = \{i | 1 + \Sigma_{\lambda=1}^{\alpha} MN_\lambda + (\beta - 1)N_{\alpha+1} \leq i \leq \Sigma_{\lambda=1}^{\alpha} MN_\lambda + \beta N_{\alpha+1}\}$. Notice that the $(\alpha + 1)$th component of $\Sigma_{i \in S_j} v(i)$ is equal to 1 and all the other components of $\Sigma_{i=S_j} v(i)$ are equal to $N_{\alpha+1}\,\varepsilon$ which is less than 1 by the selection of ε.

For convenience, let

$$V_{i,j} = \{p | 1 + \sum_{\lambda=1}^{j-1} MN_\lambda + (i-1)(N_j - 1) \leq p \leq \sum_{\lambda=1}^{j-1} MN_\lambda + i(N_j - 1)\},$$

where $i \in \mathbf{M}$ and $j \in \mathbf{k}$. Furthermore, for $j \in \mathbf{k}$, let

$$W_j = \{q | 1 + \sum_{\lambda=1}^{j-1} MN_\lambda + M(N_j - 1) \leq q \leq \sum_{\lambda=1}^{j} MN_j\}.$$

Consider the partition $P' = [S_1', \cdots, S_{M+1}']$, where

$$S_i' = \begin{cases} \bigcup_{j=1}^{k} V_{i,j} & \text{if } i \neq M + 1, \text{ and} \\ \bigcup_{j=1}^{k} W_j & \text{if } i = M + 1. \end{cases}$$

It is easy to verify that $P' \in \mathscr{C}(v)$.

Observe that $|\Delta|/\text{opt}(v) \geq |\Delta|/|P'| = kM/(M + 1) = k - (k/M + 1)) \geq k - \delta$. □

Remark. We do not claim that the lower bound in Theorem 5 is sharp. In fact, reasoning as in Theorem 4 and using the results in [1–5], it is not hard to show that, for any $\delta > 0$, there exists some function $v:\mathbf{n} \rightarrow \theta^k$, for which the FF algorithm produces a partition $\Delta \in \mathscr{C}(v)$, with $|\Delta|/\text{opt}(v) \geq k + (7/10) - \delta$.

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