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CATEGORIES OF CHAIN-COMPLETE POSETS

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Abstract. We investigate the existence of various limits and colimits in three categories: CPI (chain-complete posets and isotone maps); CPC (chain-complete posets and chain-continuous maps); CPC* (chain-complete posets and chain-*continuous maps). Among other things we show CPC* to be complete and cocomplete. By way of contrast we show that LC* (complete lattices and chain-*continuous maps) is neither complete nor cocomplete. We also introduce a construction which yields the chain-completion of a poset and other "completions" as special cases.

1. Introduction

Chain-complete posets are posets in which every chain, including the empty chain, has a sup. Chain-complete posets have many of the properties of complete lattices (including the existence of fixpoints). See Markowsky [7] for a detailed description of the properties of chain-complete posets. Many authors have used complete lattices, chain-complete posets and other closely related ordered structures in the theory of computation (e.g. see [2–6, 9–13]).

In this paper, we investigate the constructions (in the categorical sense) which are possible with chain-complete posets. We show that under identical circumstances it is *impossible* to carry out these same constructions using complete lattices.

We investigate in detail three categories: CPI (chain-complete posets and isotone maps); CPC (chain-complete posets and chain-continuous maps, i.e., maps that preserve sups of *nonempty* chains); CPC* (chain-complete posets and chain-*continuous maps, i.e., maps that preserve arbitrary sups of chains). Among other things, we show CPC* to be both complete and cocomplete in the sense of Mitchell [8]. By way of contrast, we show that the category LC* (complete lattices and chain-*continuous maps) is neither complete nor cocomplete, lacking sums and equalizers (see Theorem 2.8).

For additional terminology see Birkhoff [1] and Markowsky [7]. Our main results are summarized in Table 1.

Table 1

	CPI	CPC	CPC*
Direct limits	No	No	Yes
Inverse limits	Yes	Yes	Yes
Null objects	Yes	Yes	Yes
Conull objects	No	No	Yes
Zero objects	No	No	Yes
Nonempty sums	No	No	Yes
Nonempty products	Yes	Yes	Yes
Pushouts	No	No	Yes
Pullbacks	No	No	Yes
Coequalizers	No	Yes	Yes
Equalizers	No	No	Yes
Cocomplete	No	No	Yes
Complete	No	No	Yes
Projectives	Yes	Yes	Yes
Injectives	No	No	No

If we let LI (LC) be the full subcategory of CPI (CPC) whose objects are complete lattices, we find that LI (LC) inherits many of the properties of CPI (CPC). Furthermore, many of the same proofs are valid in LI (LC). However, the corresponding full subcategory of CPC*, LC* is neither complete nor cocomplete (see Theorem 2.8).

In Theorem 2.4, we introduce a construction which generalizes the chain-completion introduced in [7].

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2. Main results

Notation. Let P be a poset (posets are nonempty by definition). If P has a least element, we denote it by O. If n is an integer, we use n to denote the set $\{0, 1, \ldots, n-1\}$ ordered in the usual way.

Let P and Q be posets, then $P + Q$ ($P \times Q$, $P \Theta Q$), the *cardinal sum* (*cardinal product, ordinal sum*) of P and Q, is the poset consisting of the disjoint union (Cartesian product; disjoint union) of P and Q and ordered as follows: $a \leq b$ if and only if a and b both belong to P or Q and in P or Q, $a \leq b$ ($(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$; $a \leq b$ if $a \leq b$ in $P + Q$ or $a \in P$ and $b \in Q$). For more details on these operations see [1].

Remark. Observe that a nonempty Cartesian product of chain-complete posets is also chain-complete with respect to the componentwise ordering. In fact, it is easy to see that the Cartesian product together with the usual projection maps is the product in CPI, CPC and CPC*. Finally, it is easy to see that the one element poset is a null object in CPI, CPC, and CPC*.

Theorem 2.1. *CPI and CPC lack conull objects (and hence zero objects), nonempty sums, direct limits, equalizers, pushouts, pullbacks, and injectives.*

Proof. Let P be any chain-complete poset. Then $P \times 2$ is also chain-complete and the maps $f_i : P \to P \times 2$ ($i \in 2$) given by $f_i(a) = (a, i)$ are continuous and distinct. Hence CPI and CPC lack conull objects.

We now show that sums and direct limits need not exist in general in CPI. Similar arguments apply to CPC.

Let $P_1 = \{a\}$, $P_2 = \{b\}$, $P_3 = \{c\} \Theta (\{d\} + \{e\}) \Theta \{f\}$, and $P_4 = \{g\} \Theta P_3$. Suppose P_1 and P_2 had a sum $(X, \{i_1, i_2\})$ with $i_j : P_j \to X$ ($j = 1, 2$) isotone maps. Let $f_1 : P_1 \to P_3$ and $f_2 : P_2 \to P_3$ be given by $f_1(a) = d$ and $f_2(b) = e$. There exists a unique isotone map $h : X \to P_3$, such that $h \circ i_j = f_j$ for $j = 1, 2$. Note that $h(O_X) = c$. Let $i : P_3 \to P_4$ be the obvious inclusion and $\varphi_3 : P_3 \to P_4$ be given by $\varphi_3(x) = x$ if $x \neq c$ and $\varphi_3(c) = g$. Let $\varphi_j = i \circ f_j$ for $j = 1, 2$ and observe that for $h_1 = i \circ h$ and $h_2 = \varphi_3 \circ h$, $h_k \circ i_j = \varphi_j$ for $j, k = 1, 2$, but $h_1 \neq h_2$. Thus X cannot be the sum of P_1 and P_2 since there must exist a *unique* isotone map $h' : X \to P_4$ such that $h' \circ i_j = \varphi_j$ for $j = 1, 2$.

Consider the following direct family of CPI: $F = (\{P_i\}_{i \geq 0}, \{f_{i,j}\}_{i \leq j})$, where $P_i = \{-i\} \Theta \{-i+1\} \Theta \cdots \Theta \{0\} \Theta \{1\} \cdots \Theta \{i\}$, and $f_{i,j} : P_i \to P_j$ is just the natural inclusion. Let $Q_1 = \{-\infty\} \Theta \mathbf{Z} \Theta \{\infty\}$ and $Q_2 = \{-\infty'\} \Theta Q_1$, where $\mathbf{Z}$ is the set of all integers with the usual ordering. The natural inclusion maps $g_n : P_n \to Q_1$ and $h_n : P_n \to Q_2$ form a compatible family with F. If a direct limit $(X, \{i_n\})$ existed for F, there would exist *unique* maps $\varphi : X \to Q_1$ and $\theta : X \to Q_2$ such that $\varphi \circ i = g_n$ and $\theta \circ i_n = h_n$ for all n. Observe that $\varphi(O_X) = -\infty$. Let $\rho_1 : Q_1 \to Q_2$ be the natural inclusion map and $\rho_2 : Q_1 \to Q_2$ be given by $\rho_2(x) = x$ if $x \neq -\infty$ and $\rho_2(-\infty) = -\infty'$. It is clear that for $\theta_i = \rho_i \circ \varphi$ ($i = 1, 2$) we have $\theta \circ i_n = h_n$ contradicting the uniqueness of θ, since $\theta_1 \neq \theta_2$.

We now show that CPI and CPC lack equalizers. Let $P_1 = \{a\}$ and $P_2 = \{b\} \Theta (\{c\} + \{d\}) \Theta \{e\}$ and $f_1, f_2 : P_1 \to P_2$ given by $f_1(a) = c$ and $f_2(a) = d$. Clearly, a limit for this diagram does not exist since for any poset (more generally any nonempty set) S there cannot be a map $g : S \to P_1$ such that $f_1 \circ g = f_2 \circ g$.

The following example shows that CPI and CPC lack pullbacks. Let $P_1 = P_2 = 1$ and $A = \{a\} \Theta (\{b\} + \{c\}) \Theta \{d\}$. Let $f_1 : P_1 \to A$ be given by $f_1(O) = b$ and $f_2 : P_2 \to A$ given by $f_2(O) = c$. Clearly, for all posets X and maps $g_i : X \to P_i$ ($i = 1, 2$) $f_i \circ g_i \neq f_{3-i} \circ g_{3-i}$.

The following example shows that CPI and CPC lack pushouts. Let $B = 1$, $P_1 = P_2 = A$ of the preceding paragraph. Let $f_i : B \to P_i$ ($i = 1, 2$) be given by $f_i(O) = b$. Suppose Z was the pushout with maps $g_i : P_i \to Z$. Consider the chain-complete poset C, whose diagram looks like Fig. 1 and D which is $1 \Theta C$. Let

$h_i : P_i \to C$ be given by $h_i(b) = b$, otherwise $h_i(x) = x_i$. There is a unique map $h : Z \to C$ such that $h \circ g_i = h_i$. Let $i_1 : C \to D$ be the obvious inclusion and $i_2 : C \to D$ the map given by $i_2(x) = i_1(x)$ if $x \neq O$ and $i_2(O) = O_D$. Observe that since Z is chain-complete (in particular Z has a least element), h is a surjection. Now $i_1 \circ h_1 \circ f_1 = i_1 \circ h_2 \circ f_2$. But the distinct maps $i_1 \circ h$ and $i_2 \circ h$ when composed with g_i yield $i_1 \circ h_i$, contradicting the supposition that Z is the pushout.

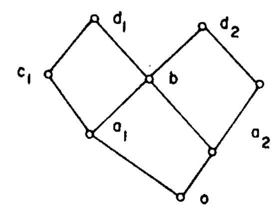

Fig. 1

In CPI and CPC the monomorphisms are exacly the injective maps. If CPI and CPC had injectives (i.e., for every chain-complete poset P, there is a monomorphism of P into an injective chain-complete poset) there would be a chain-complete injective poset Q containing at least two elements. Let $A' = \{a\} \Theta (\{b\} + \{c\})$, and let $A = 3$ with the usual ordering. Let $f : A' \to A$ be given by $f(a) = 0$, $f(b) = 2$, and $f(c) = 1$. Let $g : A' \to Q$ be given by $g(a) = g(b) = O_Q$ and $g(c) = x$, where x is any nonzero element of Q. Clearly there does not exist any isotone map $h : A \to Q$ such that $h \circ f = g$, contradicting the assumption that Q was injective. □

Remark. Since CPI and CPC lack some limits and colimits (e.g., pullbacks and pushouts), they are neither complete nor cocomplete.

Corollary. *CPC* lacks injectives.*

Proof. The proof is identical with that used in Theorem 2.1 for CPI and CPC. □

Remark. It is easy to see that projectives exist in CPI, CPC and CPC*. The only projective objects are posets of the form $1 \Theta X$ where X is a totally unordered set, i.e., the elements of X are mutually incomparable. The details are straightforward and left to the reader.

Theorem 2.2. *CPI and CPC have inverse limits.*

Proof. We will show that CPI has inverse limits, but since the proof for CPC is similar we will omit it. Let D be a directed set and $F = (\{P_a \mid a \in D\}, \{f_{b,a} \mid a, b \in D,\ b \geq a\})$ be an inverse family (in particular

$f_{a,a} : P_a \to P_a$ is just the identity). Let $P = \prod_{a \in D} P_a$, and $\pi_a : P \to P_a$ the usual projection map for each $a \in D$. P is a complete poset.

Let $Q = \{x \in P \mid \text{for all } a, b \in D \text{ with } b \geq a,\ \pi_a(x) = f_{b,a}(\pi_b(x))\}$. We claim that $(Q, \{\pi_a \mid Q\}_{a \in D})$ is the inverse limit of F. Once we show that Q is a chain-complete poset, it follows quite easily that we have actually constructed the inverse limit.

Let $S = \{x \in P \mid \text{for } a, b \in D \text{ with } b \geq a,\ f_{b,a}(\pi_b(x)) \geq \pi_a(x)\}$. $S \neq \phi$, since $O_P \in S$. Let $C = \{x_\delta\}_{\delta \in \Delta} \subset S$ be a chain, and $y = \sup_P C$. Let $a, b \in D$ and $b \geq a$. Then $f_{b,a}(\pi_b(y)) \geq \sup\{f_{b,a}(\pi_b(x_\delta)) \mid \delta \in \Delta\} \geq \sup\{\pi_a(x_\delta) \mid \delta \in \Delta\} = \pi_a(y)$. Thus $y \in S$ and S is a chain-complete poset.

Let $g : S \to S$ be given by $\pi_a(g(x)) = \sup\{f_{b,a}(\pi_b(x)) \mid b \geq a\}$. The set $\{f_{b,a}(\pi_b(x)) \mid b \geq a\}$ is a directed subset of P_a, and has a sup by Corollary 2 of Theorem 1 in [7]. We now show that g is well-defined, i.e., that $g(S) \subset S$.

For $b \geq a$, $f_{b,a}(\pi_b(g(x))) = f_{b,a}(\sup\{f_{c,b}(\pi_c(x)) \mid c \geq b\}) \geq \sup\{f_{c,a}(\pi_c(x)) \mid c \geq b\} = \sup\{f_{c,a}(\pi_c(x)) \mid c \geq a\} = \pi_a(g(x))$, since the set $\{c \mid c \geq b\}$ is a cofinal subset of $\{c \mid c \geq a\}$.

Clearly g is isotone. Let Γ be the set of fixpoints of g. By Theorem 9 of [7], Γ is a chain-complete poset. We claim that $Q = \Gamma$. Clearly $Q \subset \Gamma$. If $x \in \Gamma$, then for all $a, b \in D$ with $b \geq a$, $f_{b,a}(\pi_b(x)) \geq \pi_a(x)$, while $\pi_a(x) = \sup\{f_{b,a}(\pi_b(x)) \mid b \geq a\}$, i.e., $f_{b,a}(\pi_b(x)) = \pi_a(x)$. Thus $Q = \Gamma$, and Q is a chain-complete poset. □

Remark. We note here that in the category of all posets with all isotone maps, inverse limits *do not exist*, while direct limits, sums and products do exist.

Theorem 2.3. *CPI lacks coequalizers.*

Proof. Let A be the poset whose Hasse diagram is given in Fig. 2, and B the poset whose Hasse diagram is given in Fig. 3.

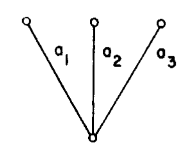

Fig. 2.

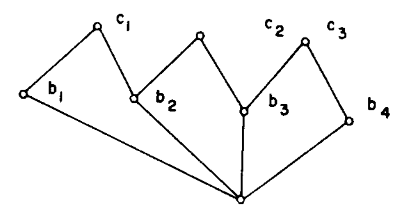

Fig. 3.

Let $f_1 : A \to B$ be given by $f_1(a_i) = b_i$ and $f_2 : A \to B$ be given by $f_2(a_1) = b_1$ and $f_2(a_i) = c_{i-1}$ for $i \geq 2$. Suppose the chain-complete poset X together with the isotone map $g : B \to X$ is a coequalizer. Let $C = \mathbf{N} \Theta \{e\}$, where $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the natural numbers ordered in the usual manner, and $D = C \Theta \{e'\}$. The map $h : B \to C$ given by $h(O) = O$, $h(b_1) = 1$, $h(b_i) = h(c_{i-1}) = i$ for all $i \geq 2$ is such that $h \circ f_1 = h \circ f_2$. Thus there must be a unique map $k : X \to C$ such that $k \circ g = h$. Since X is chain-complete, k is a surjection. Let $m_1, m_2 : C \to D$ be isotone maps such that $m_i(j) = j$ for all j, but $m_1(e) = e$ and $m_2(e) = e'$. Observe that $m_1 \circ h \circ f_1 = m_1 \circ h \circ f_2$, but that $m_1 \circ k$ and $m_2 \circ k$ are two distinct maps which when composed with g yield $m_1 \circ h$. This contradicts the supposition that X was a coequalizer. □

Remark. We will show that CPC has coequalizers. To prove this we introduce a construction modeled on the construction of the chain-completion in [7]. We will also use this construction to prove that CPC* is cocomplete.

Theorem 2.4. *Let A and B be posets and $f : A \to B$ be isotone.*

(a) *There exists a chain-complete poset B_{f^*} and isotone map $g : B \to B_{f^*}$ such that $g \circ f$ is chain-*continuous and for all chain-complete posets H and isotone maps $\alpha_1 : A \to H$, $\alpha_2 : B \to H$ such that α_1 is chain-*continuous and $\alpha_1 = \alpha_2 \circ f$, there is a unique chain-*continuous map $h : B_{f^*} \to H$ such that $h \circ = g = \alpha_2$.*

(b) *If B has a least element, there exists a chain-complete poset B_{f^0} and isotone map $g : B \to B_{f^0}$ such that $g \circ f$ is chain-continuous and for all chain-complete posets H and isotone maps $\alpha_1 : A \to H$, $\alpha_2 : B \to H$ such that α_1 is chain-continuous and $\alpha_1 = \alpha_2 \circ f$, there is a unique chain-continuous map $h : B_{f^0} \to H$ such that $h \circ g = \alpha_2$.*

Note that B_{f^*} and B_{f^0} are unique up to up to isomorphism.

Proof. (a) The proof depends heavily on techniques developed in [7] and is almost identical in structure to the proof of Theorem 6 in [7] so we will just sketch the proof. All terms not defined in this paper can be found there.

We define a closure operation γ on the subsets of B as follows. If $S \subset B$, we set $S_0 = \{y \in B \mid \text{for some } x \in S,\ y \leq x\}$. For any ordinal $\alpha > 0$, we let $S_\alpha = \{y \in B \mid \text{for some chain } C \subset A \text{ such that } a = \sup_A C, f(C) \subset \bigcup_{\lambda<\alpha} S_\lambda \text{ and } y \leq f(a)\} \cup (\bigcup_{\lambda<\alpha} S_\lambda)$. We let $\gamma(S) = S_\alpha$, where α is the least ordinal α such that $S_\alpha = S_{\alpha+1}$. Then $B_{f^*} = \gamma^*(\{C \subset B \mid C \text{ is a chain}\})$ in the notation of Definition 4 [7].

The function $g : B \to B_{f^*}$ is given by $g(a) = \gamma(\{a\})$, while $h : B_{f^*} \to H$ is given by $h(S) = \sup_H \alpha_2(S)$. The verification of the remaining properties asserted in (a) proceeds according to the lines laid out in Theorem 6 in [7].

(b) This can be proven as above by systematically disallowing the empty set. However, (b) can be made to follow from (a) as follows. Let $A' = \{a\} \Theta A$, $B' = \{b\} \Theta B$ ($a \notin A, b \notin B$) and $f' : A' \to B'$ be given by $f'(a) = b$ and $f' \mid A = f$. It is easy to see that $(B')_{f^*}$ has the form $\{c\} \Theta Q$ ($c \notin Q$), where Q is chain-complete. Let $g' : B' \to (B')_{f^*}$ be the map of (a) and $g : B \to Q$ be given by $g = g' \mid B$. It is easy to see that g together with Q has the property required in (b). □

Remark. Various instances of Theorem 2.4 are of special interest. Let $A = B$ and f be the identity. Then A_{f^*} is the chain-completion $\bar{A}$ of A (see Definition 6 in [7]). Let B be a poset and A be the poset consisting of the elements of B totally *unordered*. Let f be the set-theoretical identity. Then B_{f^*} has the property that any isotone map of B into a chain-complete poset H extends uniquely to a chain-*continuous map from B_{f^*} into H. One can construct other variations along the lines suggested in [7].

Theorem 2.5. *CPC has coequalizers.*

Proof. Let $f, g : P \to Q$ be two chain-continuous maps, where P and Q are both chain-complete posets.

We define a relation $\sim$ on Q as follows. $a \sim b$ iff there exists $a_1, a_2, \ldots, a_k \in Q$ and $x_1, \ldots, x_{k-1} \in P$ such that $a_1 = a$, $a_k = b$ and $\{a_i, a_{i+1}\} \subset \{f(x_i), g(x_i)\}$. It is easy to see that $\sim$ is an equivalence relation on Q.

For all $a \in Q$, we let $[a] = \{b \in Q \mid a \sim b\}$. Consider the relation R defined on $Q/\sim\ = \{[a] \mid a \in Q\}$, by $[a] \leq [b]$ if and only if there exist $a_1, \ldots, a_{2k} \in Q$ such that $a \sim a_1$, $b \sim a_{2k}$, $a_{2i} \sim a_{2i+1}$ and $a_{2i+1} \leq a_{2i+2}$. Clearly R is reflexive and transitive. We introduce a relation $\sim^*$ on $Q/\sim$ by $[a] \sim^* [b]$ iff $[a]R(b)$ and $[b]R[a]$. Clearly, $\sim^*$ is an equivalence relation.

For each $a \in Q$, we let $[[a]] = \{[b] \in Q/\sim \mid [a] \sim^* [b]\}$. Let ${}_1Q = \{[[a]] \mid a \in Q\}$. It is easy to see that ${}_1Q$ is a poset with $[[a]] \leq [[b]]$ iff $[a]R[b]$. Let $\pi : Q \to {}_1Q$ be given by $\pi(a) = [[a]]$. Note that π is isotone.

Observe that for all $x \in P$, $f(x) \sim g(x)$, i.e., $[f(x)] = [g(x)]$, and thus $\pi \circ f = \pi \circ g$. Furthermore, $\pi(O_Q)$ is the least element of ${}_1Q$.

Suppose $h : Q \to T$ is chain-continuous (T is chain-complete) and $h \circ g = h \circ f$. We claim there exists a *unique isotone* map $\bar{h} : {}_1Q \to R$ such that $\bar{h} \circ \pi = h$. Since π is surjective, $\bar{h}$ is unique since $\bar{h}([[a]])$ must equal $h(a)$. We thus need to show that h is isotone and well-defined.

Consider the map $\rho : Q/\sim\ \to T$ given by $\rho([a]) = h(a)$. ρ is well defined since, if $a \sim b$, then there exist $a_1, \ldots, a_k \in Q$ and $x_1, \ldots, x_{k-1} \in P$ such that $\{a_i, a_{i+1}\} \subset \{f(x_i), g(x_i)\}$. Since $h \circ f = h \circ g$, $h(a_i) = h(a_{i+1})$ for all i, i.e., $h(a) = h(b)$. Thus ρ is well-defined. Note that if $[a]R[b]$, then $\rho([a]) \leq \rho([b])$. Thus $[a] \sim^* [b]$ implies that $\rho([a]) = \rho([b])$, i.e., if $[[a]] = [[b]]$, $h(a) = h(b)$. Thus $\bar{h}$ is well-defined. If $[[a]] \leq [[b]]$, $[a]R[b]$ and $h(a) \leq h(b)$. Thus $\bar{h}$ is isotone.

We claim that $(k \circ \pi, {}_1Q^0_\pi)$ (where ${}_1Q^0_\pi$ and $k : {}_1Q \to {}_1Q^0_\pi$ are as in Theorem 2.4(b)) is a coequalizer for $f, g : P \to Q$. Given any chain-complete poset T and chain-coninuous map $\alpha_1 : Q \to T$ such that $\alpha_1 \circ f = \alpha_1 \circ g$, there is a unique isotone $\alpha_2 : {}_1Q \to T$ such that $\alpha_2 \circ \pi = \alpha_1$. From Theorem 2.4(b) it follows that there exists a *unique* chain-continuous $\alpha_3 : {}_1Q_f^0 \to H$ such that $\alpha_3 \circ k = \alpha_2$. In particular, $\alpha_3 \circ k \circ \pi = \alpha_1$. At this point, however, there is a subtlety that must not be glossed over. To conclude that $(k \circ \pi, {}_1Q_\pi^0)$ is the coequalizer, we must show that for any chain-continuous $\beta : {}_1Q_\pi^0 \to T$ such that $\alpha_1 = \beta \circ k \circ \pi$, $\beta = \alpha_3$. Since $\beta \circ k \circ \pi = \alpha_3 \circ k \circ \pi$, and π is *surjective*, $\beta \circ k = \alpha_3 \circ k = \alpha_2$. By uniqueness of α_3 we can conclude that $\beta = \alpha_3$. □

Remark. We will now show that CPC* is complete and cocomplete. These results will automatically supply the remaining entries for the table presented at the beginning.

Theorem 2.6. *CPC* is complete.*

Proof. Final objects in CPC* are just the singleton posets. Thus arbitrary products exist, since for nonempty families products are just Cartesian products with the usual projections. Suppose we have a diagram in CPC*, which we will represent as $F = (\{P_\alpha \mid \alpha \in \Delta\}, \{f_\beta \mid \beta \in \Gamma\}, \rho, \delta)$ where $\delta, \rho : \Gamma \to \Delta$ are such that $f_\beta : P_{\delta(\beta)} \to P_{\rho(\beta)}$ is chain-*continuous, i.e., $\delta(\beta)$ locates the domain of f_β, while $\rho(\beta)$ locates the codomain of f_β. We claim that the limit for F is $R = (L, \{\pi_\alpha \mid L\}_{\alpha \in \Delta})$ where $L = \{x \in \Pi_{\alpha \in \Delta} P_\alpha \mid \pi_{\rho(\beta)}(x) = f_\beta(\pi_{\delta(\beta)}(x))$ for all $\beta \in \Gamma\}$ with $\pi_\alpha : \Pi_{\alpha \in \Delta} P_\alpha \to P_\Delta$ being the usual projection map for each $\gamma \in \Delta$. Since each f_β is chain-*continuous, $L \neq \emptyset$, since the zero-element of $\Pi_{\alpha \in \Delta} P_\alpha$ is in L. It is easy to see that if $C \subset L$ is a chain, then $\sup_{\Pi P_\alpha} C \in L$. Thus L is a chain-complete poset. It is now straightforward to show that R is a limit for F. □

Theorem 2.7. *CPC* is cocomplete.*

Proof. The initial objects (empty sums) are simply the singleton posets (recall that all mappings are chain-*continuous). The sum of a nonempty family is just the disjoint union, with the least elements of each component identified, together with the obvious inclusion maps. Formally, if $\{P_\alpha\}_{\alpha \in \Delta}$ is the family in question, the sum is $(P, \{i_\alpha\})$ where $P = O_P \Theta (\Sigma_{\alpha \in \Delta}(P_\alpha - \{O_{P_\alpha}\}) \times \{\alpha\}))$ and $i_\alpha : P_\alpha \to P$ is given by $i_\alpha(x) = (x, \alpha)$ if $x \neq O_{P_\alpha}$ and $i_\alpha(O_{P_\alpha}) = O_P$.

Let $F = (\{P_\alpha\}_{\alpha \in \Delta}, \{f_{\beta \in \Gamma}\}, \rho, \delta)$ be a diagram in CPC*, in the notation of Theorem 2.6. Let $(P, \{i_\alpha\})$ be the sum constructed above for the family $\{P_\alpha\}$. On P we define an equivalence relationship $\sim$ as follows. For $(x, \alpha_1), (y, \alpha_2) \in P$ we say $(x, \alpha_1) R (y, \alpha_2)$ if and only if $(x, \alpha_1) = (y, \alpha_2)$ or there exists $\beta \in \Gamma$ such that $\delta(\beta) = \alpha_1$, $\rho(\beta) = \alpha_2$ and $f_\beta(x) = y$. We say xLy if and only if yRx. For $x, y \in P$ we say $x \sim y$ if and only if there exists a finite sequence $x_1, \ldots, x_k \in P$ such that $x = x_1$, $y = x_k$ and for all $1 \leq i \leq k-1$, either $x_i L x_{i+1}$ or $x_i R x_{i+1}$. Let $Q = \{[x] \mid x \in P\}$ where $[x]$ is the equivalence class of x with respect to $\sim$. For $x, y \in P$, define $[x] \leq_1 [y]$ to mean that there exist $s \in [x]$, $t \in [y]$ such that $s \leq t$. Now define $[x] \leq_2 [y]$ to mean that there exist $x_1, \ldots, x_k \in P$ such that $[x] = [x_1]$, $[y] = [x_k]$ and for all $1 \leq i \leq k-1$, $[x_i] \leq_1 [x_{i+1}]$. Clearly, $\leq_2$ quasiorders Q. Finally, we define another equivalence relationship $\sim'$ on P, i.e., $x \sim' y$ if and only if $[x] \leq_2 [y]$ and $[y] \leq_2 [x]$. Let $R = \{[x]' \mid x \in P\}$ where $[x]'$ is the equivalence class of x with respect to $\sim'$. R is a poset with $[x]' \leq [y]'$ meaning $[x] \leq_2 [y]$. In general, R is not complete.

Let $f : P \to R$, be given by $f(x) = [x]'$. Let R_{f^+} and $g : R \to R_{f^+}$ be as described in Theorem 2.4(a). We claim that the colimit for F is $(R_{f^+}, \{g \circ f \circ i_\alpha \mid \alpha \in \Delta\})$. Observe first that $g \circ f \circ i_\alpha$ is chain-*continuous. If $\beta \in \Gamma$ and $x \in P_{\delta(\beta)}$, then $(x, \delta(\beta)) R (f_\beta(x), \delta(\beta))$, i.e., $(x, \delta(\beta)) \sim (f_\beta(x), \rho(\beta))$. Thus $g \circ f \circ i_{\delta(\beta)} = g \circ f \circ i_{\rho(\beta)} \circ f_\beta$.

Suppose we are given any family $(T, \{h_\alpha\}_{\alpha \in \Delta})$ which is cocompatible with F. Since $(P, \{i_\alpha\})$ is the sum of $\{P_\alpha\}_{\alpha \in \Delta}$, there exists a unique chain-*continuous $h : P \to T$ such that $h_\alpha = h \circ i_\alpha$ for each $\alpha \in \Gamma$. We claim the map $h' : R \to T$ given by $h'([x]') = h(x)$ is well-defined and isotone. Clearly, $x \sim y$ implies that $h(x) = h(y)$, and $[x] \leq_2 [y]$ implies that $h(x) \leq h(y)$. In short, h' is isotone and $h' \circ f = h$. Thus there exists a *unique* chain-*continous map $h'' : R_{f^+} \to T$ such that $h'' \circ g = h'$. In particular, $h'' \circ g \circ f \circ i_\alpha = h' \circ f \circ i_\alpha = h \circ i_\alpha = h_\alpha$ for all α. To conclude that CPC* is cocomplete we must show that $k = h''$. Since P is the sum of $\{P_\alpha\}$, $k \circ g \circ f = h = h'' \circ g \circ f$. Since f is surjective, $k \circ g = h'' \circ g$, which by the uniqueness of h'' implies that $h'' = k$. □

Remark. The following theorem shows that in general, chain-*continuous maps do not preserve lattice structure under limits and colimits. Recall that LC* is the full category of CPC* whose objects are all complete lattices.

Theorem 2.8. *LC* lacks sums and equalizers, i.e., LC* is neither complete nor cocomplete.*

Proof. Let $L_1 = L_2 = 2$ ordered as usual. Suppose $(X, \{f_1, f_2\})$ with $f_i : L_i \to X$ is the sum. Let $L_3 = (\{a\} \Theta (\{b\} + \{c\})) \Theta \{d\}$ and $L_4 = L_3 \Theta \{e\}$.

Let $g_1 : L_3 \to L_4$ be the natural inclusion and $g_2 : L_3 \to L_4$ be given by $g_2(x) = x$ if $x \neq d$, while $g_2(d) = e$. Finally let $h_1 : L_1 \to L_3$ and $h_2 : L_2 \to L_3$ be given by $h_1(O) = h_2(O) = a$, $h_1(1) = b$, $h_2(1) = c$. Let $h : X \to L_3$ be the unique map such that $h \circ f_i = h_i$. Note that h is surjective since $h(f_1(1) \vee_X f_2(1)) = d$. Let $h_i' : L_i \to L_4$ $(i = 1, 2)$ be given by $h_i' = g_1 \circ h_i$. Clearly, the two *distinct* maps $g_1 \circ h$ and $g_2 \circ h$ when composed with f_i yield h_i', contradicting the supposition that $(X, \{f_1, f_2\})$ was the sum of L_1 and L_2.

Let L_1 be the lattice corresponding to the Hasse diagram shown in Fig. 4 and L_2 the lattice corresponding to the Hasse diagram shown in Fig. 5. Let $f_1 : L_1 \to L_2$ and $f_2 : L_1 \to L_2$ be given by $f_1(0) = f_2(0) = 0$, $f_1(a) = f_2(a) = U$, $f_1(b) = f_2(b) = V$, $f_1(c) =$

W, $f_2(c) = X$, $f_1(d) = f_2(d) = Z$, $f_1(e) = f_2(e) = Y$ and $f_1(1) = f_2(1) = 1$. Let $(X, \{g\})$ with $g : X \to L_1$ be the equalizer for $(\{L_1, L_2\}, \{f_1, f_2\})$.

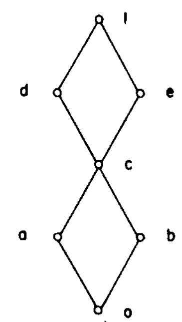

Fig. 4.

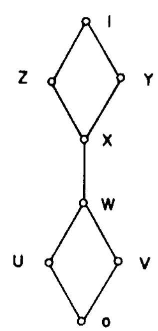

Fig. 5.

We claim that $g(X) = \{0, a, b, d, e, 1\}$. Clearly, $g(X) \subset \{0, a, b, d, e, 1\} = B$, since $f_1 \circ g = f_2 \circ g$. For $t \in B$, let $h : 2 \to L_1$ (2 is ordered as usual) be given by $h(0) = 0$ and $h(1) = t$. Since there exists $h' : 2 \to X$ such that $g \circ h' = h$, $t \in g(X)$.

We claim that g is injective. Suppose for some $t \in B$, $g^{-1}(t) \supset \{y_1, y_2\}$ with $y_1 \neq y_2$. Let $h : 2 \to L_1$ be given by $h(0) = 0$ and $h(1) = t$. Let $h'_i : 2 \to X$ $(i = 1, 2)$ be given by $h'_i(0) = 0$ and $h'_i(1) = y_i$. Thus we get two distinct maps which composed with g yield h, contradicting the supposition that X is an equalizer.

Thus X consists of six elements $\{0_x, a_x, b_x, d_x, e_x, 1_x\}$, where $g(t_x) = t$. Since g is isotone $a_x \not\leq b_x \not\leq a_x$ and $d_x \not\leq e_x \not\leq d_x$.

Let L_3 be as earlier in the proof and $h : L_3 \to L_1$ given by $h(a) = 0$, $h(b) = a$, $h(c) = b$ and $h(d) = d$. Since $f_1 \circ h = f_2 \circ h$ we have an isotone map $h' : L_3 \to X$ such that $g \circ h' = h$. This implies that $d_x \geq a_x, b_x$. Similarly, $e_x \geq a_x, b_x$. Thus X has the Hasse diagram given in Fig. 6, which is impossible since X must be a lattice. □

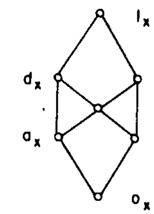

Fig. 6.

Remark. One can show by an argument similar to Theorem 2.2 that LC* has inverse limits.

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