Proc. Univ. of Houston
Lattice Theory Conf..Houston 1973

SOME COMBINATORIAL ASPECTS OF LATTICE THEORY*

George Markowsky
Harvard University
Cambridge, Mass. 02138

This paper will discuss some new lattice-theoretic constructions of combinatorial interest. Throughout, all lattices will be assumed to be finite unless the contrary is stated, and most proofs will be omitted. Proofs and generalizations (e.g. to infinite lattices) are in the author's Doctoral Thesis [13].

After a few technical preliminaries we will discuss a basic representation theorem for lattices and give some applications of it, including a new characterization of distributive lattices and some combinatorial results having to do with the representation of lattices and posets by subsets of the power set of some given set. In Part II, we introduce the poset of join-irreducible and meet-irreducible elements of a lattice, a construction which bears the same relationship to the given lattice, as the poset of join-irreducible elements bears to the corresponding finite distributive lattice. After describing the properties of the poset of join-irreducible

* This research has been partially supported by ONR Contract N00014-67-A-0298-0015.

and meet-irreducible elements, we will give some applications of this construction, including the extension of the work of Crapo and Rota [7] on the factorization of relatively complemented lattices of finite length to all lattices of finite length. We will then discuss the enumeration of the elements of the free distributive lattice on n generators, a problem first proposed by Dedekind [8] in 1897.

Much of the work in Parts I and II has been stimulated by the following question. How much of the structure of a lattice is 'recoverable' from its join-irreducible and meet-irreducible elements? As we shall see, the answer to this question is that by concentrating only on certain relations between join-irreducible and meet-irreducible elements we are able to reconstruct the whole lattice, and can obtain information about the lattice which would be difficult to obtain from the whole lattice directly, such as its factorization.

I. THE BASIC REPRESENTATION THEOREM AND APPLICATIONS

We first introduce some notation. If n is an integer, by $\underline{n}$ we shall mean $\{1, \ldots, n\}$. Of course if $n \leq 0$, $\underline{n} = \emptyset$. If X is a set, we shall denote the cardinality of X by $|X|$, and the power set of X by 2^X. Note that we shall always consider 2^X to be a lattice in the obvious way. We will use $\leq$ and $<$ for set inclusion and proper set inclusion respectively. If L is a lattice, we denote by $J(L)$ the set of all join-irreducible elements of L (recall L is finite) and by $M(L)$ the set of all the meet-irreducible elements of L. $\wedge$ and $\vee$ denote meet and join respectively.

The following representation theorem will be our starting point. It has been used by Zaretskii [18] and is closely related to the dual of the representation by principal dual ideals due to Birkhoff and Frink [2]. It can be generalized quite a bit, and was discovered by the author while he was investigating the structure of the semigroup of binary relations ([14]).

THEOREM 1. Let L be a lattice. The map $f: L \to 2^{M(L)}$ given by $f(a) = \{y \in M(L) \mid y \not\geq a\}$ is injective and join-preserving (and hence order-preserving).

Theorem 1 has a number of consequences. The following corollary is obvious even without Theorem 1.

COROLLARY Let L be a lattice $|J(L)| = j$ and $|M(L)| = k$, then the length of $L \leq \min\{k,j\}$.

The following theorem gives a new combinatorial characterization of finite distributive lattices. It is well known that (c) below implies (a) and (b). But the converse seems to be new.

THEOREM 2. Let L be a finite lattice. The following are equivalent.

(a) L has length n, satisfies the Jordan-Dedekind chain condition, has n join-irreducible elements and n meet-irreducible elements.

(b) L has n join-irreducible elements, n meet-irreducible elements, and every connected (maximal) chain between I and 0 has length n.

(c) L is distributive and has n join-irreducible elements.

<u>Proof</u>: It is easy to see that (a) and (b) are equivalent and it is well known that (c) implies (a) (see Birkhoff [1]). Thus we need only show that (a) implies (c). Let L' be the dual lattice and observe that L' also satisfies (a). From Theorem 1 it follows that we can consider L and L' to be <u>join-sublattices</u> of $2^{\underline{n}}$, where by a <u>join-sublattice</u> we mean a subset of $2^{\underline{n}}$ closed under arbitrary join (union). Any such subset is of course a lattice with join being union but the

meet of two elements is not in general the intersection.

We claim that L and L' are sublattices of $2^{\underline{n}}$ and hence distributive. Let $f: L \to L'$ be an anti-isomorphism.

Note that from (a) it follows that the height of any element of L or L' is equal to its cardinality. We now make a series of observations.

(i) $|f(x)| = n - |x|$ for all $x \in L$, since a connected chain from x to $\underline{n}$ is mapped into a connected chain from ϕ to $f(x)$.

(ii) $|f(y)| - |f(x) \wedge_{L'} f(y)| = |x| - |x \cap y|$ for all $x, y \in L$, since $|f(y)| - |f(x) \wedge_{L'} f(y)|$

$= (n - |y|) - (n - |x \cup y|)$

$= |x \cup y| - |y| = |x| - |x \cap y|$.

(iii) $|f(y)| - |f(x) \cap f(y)| = |x| - |x \wedge_L y|$ for all $x, y \in L$, since $|f(y)| - |f(x) \cap f(y)|$

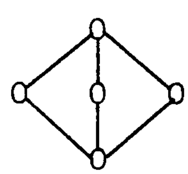
$= |f(x) \cup f(y)| - |f(x)| = (n - |x \wedge_L y|) - (n - |x|)$

$= |x| - |x \wedge_L y|$.

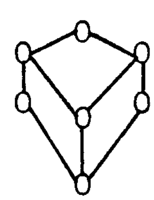
We know that $x \wedge_L y \leq x \cap y$ and $f(x) \wedge_{L'} f(y) \leq f(x) \cap f(y)$ for all $x, y \in L$. Thus $|x| - |x \wedge_L y| \geq |x| - |x \cap y|$ and $|f(y)| - |f(x) \cap f(y)| \leq |f(y)| - |f(x) \wedge_{L'} f(y)|$. But from the last inequality, (ii) and (iii) it follows that $|x| - |x \wedge_L y| \leq |x| - |x \cap y|$. Hence, $|x \wedge_L y| = |x \cap y|$ and $x \wedge_L y = x \cap y$, implying that L is a sublattice of $2^{\underline{n}}$. Similarly, L' is a sublattice.

Remark. Theorem 2 assists in identifying distributive lattices from their Hasse diagrams, since it is usually easy to identify the join-irreducible and meet-irreducible elements, as well as to determine whether a given lattice satisfies the Jordan-Dedekind chain condition. Certainly, a computer can easily be programmed to identify finite distributive lattices.

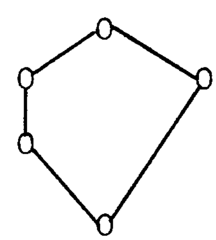
Theorem 2 can be stated as follows: a finite lattice L with n join-irreducible elements is distributive if and only if (i) it satisfies the Jordan-Dedekind chain condition, (ii) the number of meet-irreducible elements equals the number of join-irreducible elements, and (iii) the length of L is equal to the number of join-irreducible elements. The following three examples show the independence of conditions (i), (ii), and (iii).



(a)



(b)



(c)

Here n=3. (a) satisfies (i) and (ii) only, (b) (i) and (iii) only, and (c) (ii) and (iii) only.

COROLLARY 4. A finite modular lattice L is distributive if and only if its length is equal to $|J(L)|$.

Proof. It is well known that modular lattices satisfy the Jordan-Dedekind chain condition [1]. Also Dilworth has shown [1; 103] that $|J(L)|$ and $|M(L)|$ of any finite modular lattice are equal. Thus the corollary follows directly from Theorem 2 and these additional facts.

Definition. Let L be a lattice. By an embedding of L in $2^{\underline{n}}$ we mean an injective join-preserving map $f: L \to 2^{\underline{n}}$. We shall say that two embeddings f and g are distinct if $f(L) \neq g(L)$.

Theorem 1 shows that L can be embedded in $2^{\underline{n}}$ if $n \geq |M(L)|$. Actually, it is true that L can be embedded in $2^{\underline{n}}$ iff $n \geq |M(L)|$. This was first shown to be true by Zaretskii [18] and later discovered independently by the author [14]. We will not prove this result here.

An obvious question about embeddings is the following. Given a lattice L and an integer n how many distinct embeddings of L in $2^{\underline{n}}$ are there? The results above only tell us when an embedding is possible. The answer to this question follows from work done in exploring the structure of the semigroup of binary relations done by Brandon Butler, D. W. Hardy and the author [3, 4]. It can also be derived from Zaretskii's work [18]. For details about the relationship between lattices and the semigroup of binary relations see [15].

To avoid introducing too much additional theory we simply state the following theorem (which can be generalized to the case of arbitrary join-preserving maps between arbitrary complete lattices [13]).

THEOREM 3. Let L be a finite lattice such that $|L| = p$, $|M(L)| = k$. The number of distinct embeddings of L in $\underline{2}^{\underline{n}}$ is $(1/|\mathrm{Aut}\ L|) \sum_{i=0}^{k} (-1)^i \binom{k}{i} (p-i)^n$, where $\mathrm{Aut}\ L$ is the automorphism group of L. Note that the above quantity is 0 if $n < k$. A purely lattice-theoretic proof can be found in [13].

We will now consider the relationship between some of the material above, and the problem of computing the number of realizations of a given poset by a subset of $\underline{2}^{\underline{n}}$ for some integer n. By this we mean that if we are given a finite poset P, we wish to know how many subsets of $\underline{2}^{\underline{n}}$, considered as posets with inclusion being the order, there are which are isomorphic to P. This problem is treated in some detail by Hillman in [11]. We will briefly show how the theorems above apply to this problem. Our method does not always simplify the calculations involved, but it does give "geometrical" insight into the difficulties involved in representing posets by sets.

Definition. Let L be a lattice and P a poset. Then $R(L,n)$ denotes the number of ways of representing L as a join-sublattice of $2^{\underline{n}}$, $R^*(P,n)$ denotes the number of realizations of P by subsets of $2^{\underline{n}}$, and $D(P)$ denotes the distributive lattice of all closed from below subsets of P, while $i:P \to D(P)$ denotes the canonical map $i(a) = \{b \in P \mid b \leq a\}$. A subset k of L is called a meet-sublattice of L if it is closed under arbitrary meets (recall that the empty meet is always I).

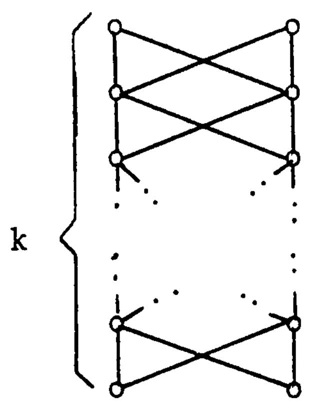
We note here that a meet-sublattice of a lattice is itself a lattice with respect to the induced order.

The key result for applying Theorem 3 to the representation of posets is the following.

THEOREM 4. Let P be a poset, and $\tilde{L}$ the set of all meet-sublattices of $D(P)$ which contain $i(P)$. Pick one representative from each isomorphism class of $\tilde{L}$, say $L_1, \ldots, L_k$. For each $i \in \underline{k}$ let $m_{L_i} = |\{Q \leq L_i \mid J(L_i) \leq Q$ and $P \simeq Q$ as posets$\}|$. Then $R^*(P,n) = \sum_{i=1}^{k} m_{L_i} R(L_i,n)$.

All of Hillman's results can be derived starting from Theorem 4, but we will not dwell on this here. Rather we will just give an example in which Theorem 4 supplies the answer more directly than any of Hillman's approaches.

EXAMPLE 1. Let P have the Hasse diagram



Then $R^*(P,n) = 2^{-k} \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (3k+1-i)^n$, since the only meet-sublattice of $D(P)$ containing $i(P)$ is $D(P)$ itself, and $|D(P)| = 3k+1$. Thus in certain cases the lattice method allows one to quickly group the essentials of the situation and arrive at the solution directly. This example illustrates the fact that Theorem 4 often allows one to see quickly how to calculate $R^*(P,n)$ and gives some idea of how complicated the calculation will be, as well as allowing one to calculate $R^*(P,n)$ for a whole class of related posets, as opposed to isolated cases. It is interesting to note that Theorem 4 shows why the poset representation problem is hard in general.

Namely, the poset representation problem involves the representation of a number of lattices which are not "obviously" related to one another. Thus we also see why the coefficients seem to vary so much in the cases that are known. However, Theorem 4 gives us enough information to describe the asymptotic behavior of $R^*(P,n)$, for a fixed P as $n\to\infty$. In particular we have the following corollaries.

COROLLARY 1. $R^*(P,n) \sim \frac{1}{|Aut(P)|}|D(P)|^n$ asymptotically as $n\to\infty$.

The following corollary is an interesting special case of Corollary 1. It tells us the number of anti-chains of size k in $2^{\underline{n}}$ and shows that as $n\to\infty$ almost every subset of $2^{\underline{n}}$ of cardinality k is an anti-chain. In the next corollary, A_k is the poset corresponding to the Hasse diagram $\underbrace{0\ 0\ 0 \ldots\ldots 0}_{k}$.

COROLLARY 2. $R^*(A_{k_1} n) \sim \frac{(2^k)^n}{k!} \sim \binom{2^n}{k}$ asymptotically (for fixed k) as $n\to\infty$.

II. THE POSET OF JOIN IRREDUCIBLE AND MEET IRREDUCIBLE ELEMENTS.

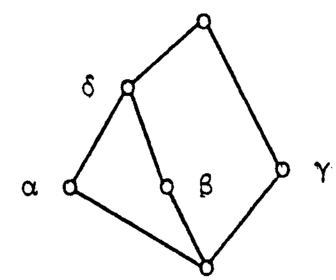
It is standard [1; p.59] that any finite distributive lattice is isomorphic with the ring of all order ideals of the partially ordered set consisting of its join-irreducible elements. Furthermore certain properties of the distributive lattice can be calculated directly from this poset of join-irreducible elements. In particular we have the following results which do not seem to have been generally considered. A proof of Theorem 5(a) can be found in [15].

THEOREM 5. Let L be a finite distributive lattice and P its poset of join-irreducible elements. Let $P = \{v_1, \ldots, v_t\}$. Then:

(a) The map $F: \mathrm{Aut}(P) \to \mathrm{Aut}(L)$ given by $F(f)(\sum_{\Delta} v_i) = \sum_{\Delta} f(v_i)$ for $\Delta \leq t$ is a group isomorphism, i.e., every element of $\mathrm{Aut}(P)$ extends naturally to an element of $\mathrm{Aut}(L)$.

(b) L is decomposable iff P is not connected and the irreducible factors of L may be gotten simply by considering the distributive lattices (i.e., the rings of closed from below subsets) associated with the connected components of P.

REMARK. Theorem 5 does not hold for arbitrary lattices. Consider the lattice L depicted by the following Hasse diagram.



$|Aut(L)| = 2$ and L is indecomposable while $|Aut(J(L))| = 6$ and $J(L)$ has 3 components.

We will now describe a poset which can be associated with all finite lattices and which has the same properties with respect to the original lattice that the poset of join-irreducibles has with respect to the corresponding distributive lattice.

<u>Definition</u>. Let L be a lattice. By $P(L)$ we mean the poset $J(L) \dot{\cup} M(L)$ (disjoint union) with the following order. Let $i_1 : J(L) \to P(L)$ and $i_2 : M(L) \to P(L)$ be the canonical injections. For $x, y \in P(L)$, $y > x$ iff (a) $y \in i_2(M(L))$, (b) $x \in i_1(J(L))$, and (c) $i_2^{-1}(y) \not\leq i_1^{-1}(x)$ in L. When talking about $P(L)$, we let $X_1 = i_1(J(L))$ and $X_2 = i_2(M(L))$. We call $P(L)$ the <u>poset of join irreducibles and meet irreducibles</u> of L or simply the <u>poset of irreducibles of L</u>.

$P(L)$ furnishes us with quite a bit of information about L. Since the proofs of the following theorems are somewhat involved we omit them and present the most important properties of $P(L)$.

THEOREM 6. Let L be a lattice and $P(L) = X_1 \cdot X_2$ its poset of irreducibles.

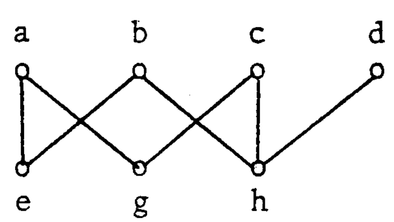
(a) Let $f: X_1 \to 2^{X_2}$ be given by $f(a) = \{b \varepsilon X_2 | b > a\}$. Then $L \simeq \Gamma_L \overset{=}{\mathrm{def}} \{\bigcup_{w \varepsilon \Delta} w | \Delta \leq f(X_1)\}$. (Thus we can reconstruct L from $P(L)$.)

(b) $\mathrm{Aut}(P(L)) \simeq \mathrm{Aut}(L)$.

(c) L is decomposable iff $P(L)$ is not connected. Futhermore, the irreducible factors of L may be gotten by applying the procedure of (a) above to each connected component of $P(L)$.

(d) For each $x \varepsilon X_2$, let $T_x = \mathrm{g.1.b.}_{\Gamma_L} S_x$ where $S_x = \{U \varepsilon f(X_1) | x \varepsilon U\}$, where Γ_L and f are as in (a). Then L is distributive iff for all $V \varepsilon f(X_1)$, $V =_{x \varepsilon V} T_x$ iff for all $x \varepsilon X_2$, $x \varepsilon T_x$. To illustrate Theorem 6 we consider the following examples.

EXAMPLE 2. Thus if we construct $P(L)$, where L is the lattice in the remark after Theorem 5, we get



$i_1(\gamma) = h, i_1(\beta) = g, i_1(\alpha) = e$,

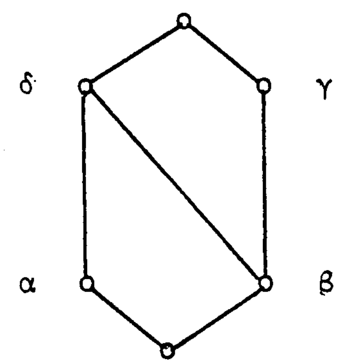
$i_2(\gamma) = a, i_2(\beta) = b, i_2(\alpha) = c$,

and $i_2(\delta) = d$.

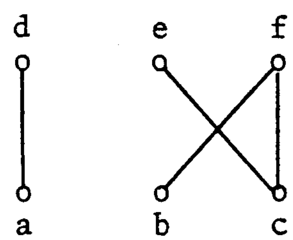
Thus $|\mathrm{Aut}(P(L))| = 2 = |\mathrm{Aut}(L)|$, L is indecomposable.

If we consider $f(L) \leq 2^{\{a,b,c,d\}}$ we see that $f(e) = \{a,b\}$, $f(g) = \{a,c\}$, $f(h) = \{b,c,d\}$. Thus $T_a = \phi$, $T_b = \phi$, $T_c = \phi$, $T_d = \{b,c,d\}$. Consequently, L is not distributive, which of course is no surprise in this case.

EXAMPLE 3. Let L have the following Hasse diagram.

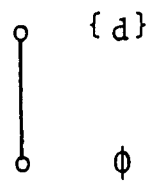


Then P(L) has the following diagram.

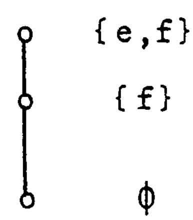


Here $i_1(\alpha) = a$, $i_2(\alpha) = f$, $i_1(\beta) = b$, $i_1(\gamma) = c$, $i_2(\gamma) = d$, $i_2(\delta) = e$.

P(L) has two components, so that P(L) has two indecomposable factors corresponding to the diagrams



and



.

Note $|Aut(P(L))| = 1 = |Aut(L)|$. Applying Theorem 6 we see that $T_d = \{d\}$, $T_e = \{e,f\}$, and $T_f = \{f\}$, and that consequently L is distributive.

Theorem 6 has some interesting consequences concerning the factorization of lattices. In particular, it leads to a simple characterization of the center of a lattice (see [1; p. 67]). The following fairly immediate corollary of Theorem 6 generalizes and extends the results described by H. Crapo and G.-C. Rota (and which follow from some work of Dilworth) for factorization of relatively complemented lattices with no infinite chains [7; Chapter 12] to the factorization of all lattices with no infinite chains.

COROLLARY. Let L be a lattice and $C(L)$ be the center of L.

(a) $x \in C(L)$ iff x is a separator of L, i.e., if $p \in J(L)$ and $q \in M(L)$ are such that $p \not\leq q$, then either $p \leq x$ or $x \leq q$.

(b) $C(L) \simeq \underline{2}^k$, where k is the number of irreducible (non-trivial) factors of L. (Note L has a unique irreducible factorization.)

(c) $L \simeq [0,c_1] \times [0,c_2] \times \ldots \times [0,c_k]$ where $c_1, \ldots, c_k$ are the points of $C(L)$.

The author is indebted to Professor Curtis Greene for suggesting that the results of [7; Chapter 12] be considered from the point of view of Theorem 6.

Before we discuss additional aspects of the poset of irreducibles we make the following definition.

<u>Definition</u>. By a bipartite digraph D, we mean a triple (X,Y,A), where X and Y are sets, $X \quad Y = \emptyset$, and $A \leq X \times Y$. A is called the set of arcs. If $S \leq X$, by $Ou(S)$ we mean $\{y \varepsilon Y |$ there exists $x \varepsilon S$ such that $(x,y) \varepsilon A\}$. Similarly, if $T \leq Y$, by $In(T)$ we mean $\{x \varepsilon X |$ there exists $y \varepsilon T$ such that $(x,y) \varepsilon A\}$. If $x \varepsilon X [y \varepsilon Y]$ we write $Ou(x)[In(y)]$ instead of $Ou(\{x\})[In(\{y\})]$. Sometimes we will use the term bidigraph to stand for bipartite digraph.

We will usually think of bidigraphs as being posets with the following ordering. If $w,z \varepsilon D$, then $w > z$ iff $w \varepsilon Y$, $z \varepsilon X$, and $w \varepsilon Ou(z)$.

From Theorem 6 we see that we can associate a "unique" bidigraph $P(L)$ to each lattice L and then recover L from $P(L)$ in a well-defined way. The following theorem shows that to any bidigraph we can associate a lattice. This theorem sets the stage for some interesting questions.

THEOREM 7. Let $D = (X,Y,A)$ be a finite bipartite digraph. Let $f: X \to 2^Y$ be given by $f(x) = Ou(x)$, and let $L_D = \{\bigcup_{w \varepsilon \Delta} w \mid \Delta \leq f(X)\}$. Then L_D is a lattice. Let $g: Y \to 2^Y$ be given by $g(y) = \text{l.u.b.}_{L_D} f(X - In(y)) = Ou(X - In(y))$. Then $f(X) \geq J(L_D)$ and $g(Y) \geq M(L_D)$.

We conclude this section by considering the following two questions. First, which finite bidigraphs are isomorphic to $P(L)$ for some lattice L? Second, suppose we are given the Hasse diagram of a finite poset, how can we determine whether or not the poset is a lattice?

The first question is answered by the following theorem.

THEOREM 8. Let $D = (X,Y,A)$ be a finite bidigraph. Then the following are equivalent.

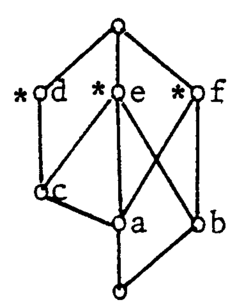
(a) $D \cong P(L)$ for some finite lattice L.

(b) For all $x \in X$, if $\Delta \leq X$ is such that $Ou(x) = Ou(\Delta)$, then $x \in \Delta$. Similarly, for all $y \in Y$, if $\Gamma \leq Y$ is such that $In(Y) = In(\Gamma)$, then $y \in \Gamma$.

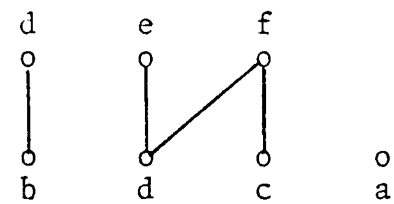
We will not answer the second question formally, but simply show how the techniques described above allow one to systematically attack the second question. The basic idea is that, given a finite P (say in the form of a Hasse diagram) one assumes that it is a lattice and constructs $P(P)$ of Theorem 6 using any element which is only covered by one element as a meet-irreducible element and any element covering only one element as a join-irreducible element. If $P(P)$ does not satisfy (b) of Theorem 8, it follows that P was not a lattice originally. If $P(P)$ does satisfy (b) of Theorem 8 one proceeds to construct $L_{P(P)}$ as in Theorem 7. From the work above, it is clear that P is a lattice iff $L_{P(P)} \cong P$. Often, it is not necessary to construct all of $L_{P(P)}$ to discover that $P \ncong L_{P(P)}$ as will be seen below.

Needless to say, if P has more than one maximal or more than one minimal element, there is no need to test it for being a lattice. Again, it is often easier to test that (b) of Theorem 8 holds for Ou and then construct $L_{P(P)}$, then to see that (b) of Theorem 8 holds for both Ou and In.

EXAMPLE 4. Let P be represented by

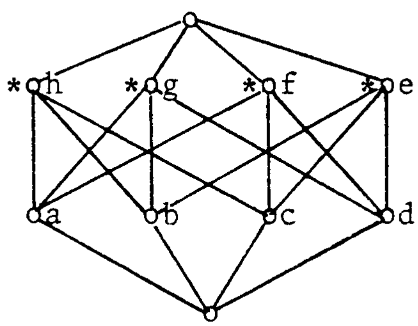


The shaded elements are the "join-irreducible" elements of P determined as above, assuming that P is a lattice. The starred elements are the meet-irreducible elements of P. We will not use i_1 and i_2 when working with P(P), in order to keep notational distractions to a minimum. Here, we have that $Ou(a) = \phi$, $Ou(b) = \{d\}$, $Ou(c) = \{f\}$, and $Ou(d) = \{e,f\}$. Since $Ou(a) = \phi$, (b) of Theorem 8 is not satisfied, since $Ou(\phi) = \phi$ and $a \notin \phi$. Hence P is not a lattice. P(P) can be represented by

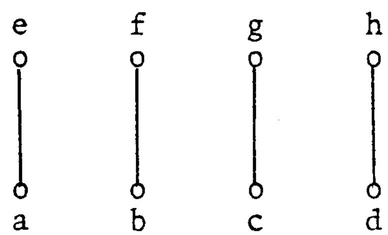


and $L_{P(P)}$ (using Theorem 7) is the lattice of Example 3.

EXAMPLE 5. Let P be represented by

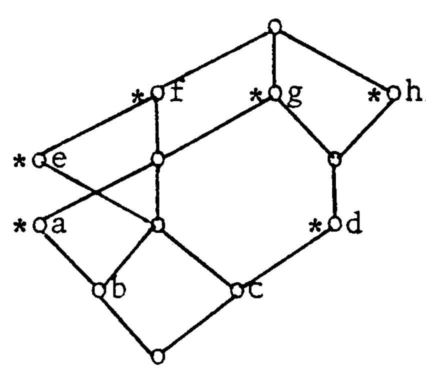


. $P(P)$ can be represented as



. Clearly $L_{P(P)} \simeq \underline{2}^{4} \ncong P$, so that P is again not a lattice.

EXAMPLE 6. Let P be represented by

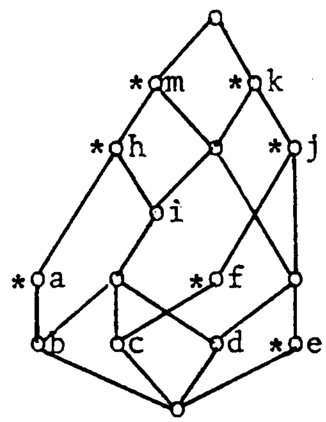


We will not draw $P(P)$, but note that the following are easily obtained from the diagram: $Ou(a) = \{d,e,h\}$, $Ou(b) = \{d\}$, $Ou(c) = \{a\}$, $Ou(d) = \{a,e,f\}$, $Ou(e) = \{a,d,g,h\}$, $Ou(h) = \{a,d,e,f,g\}$. To simplify checking whether (b) of Theorem 13 holds one should arrange the Ou's according to cardinality: $Ou(b)$, $Ou(c)$, $Ou(a)$, $Ou(d)$, $Ou(e)$, $Ou(h)$. In this way, each Ou could only be a union of preceding Ou's. $Ou(b)$ and $Ou(c)$ are singletons and thus satisfy (b). $Ou(a)$ is the first one on the list to contain an "e" or "h", while $Ou(d)$ is the first to contain an "f". "g" first appears in $Ou(e)$. "g" appears only in $Ou(e)$ and $Ou(h)$, but $Ou(e) \not\leq Ou(h)$, and hence (b) holds for all the Ou's.

Note that $In(y)$, for $y \varepsilon \{a,d,e,f,g,h\}$ is easily constructed since $In(y) = \{x \varepsilon \{a,b,c,d,e,h\} \mid y \varepsilon Ou(x)\}$. It is also easily verified that (b) holds for $In(y)$. It is easy to construct $L_{P(P)}$, and one quickly sees that $P \simeq L_{P(P)}$.

It is easy to see that $P(P)$ is connected and that therefore P is indecomposable. Furthermore, let $f \varepsilon Aut(P(P))$, it is easy to show that f = Identity, since $f(a) = a$ (a considered as belonging to X_2), $f(d) = d$ (d considered as belonging to X_2), etc.

EXAMPLE 7. Let P be represented by



Thus $Ou(c) = \{a,e\}$,

$Ou(d) = \{a,e,f\}$,

$Ou(b) = \{e,f,j\}$,

$Ou(e) = \{a,f,g,h\}$, $Ou(a) = \{e,f,g,j,k\}$, $Ou(f) = \{a,e,g,h,m\}$ $Ou(i) = \{a,e,f,g,j\}$. It is easy to see that Ou satisfies (b) of Theorem 8. However, when constructing $L_{P(P)}$, one notices almost immediately that $Ou(c) \leq Ou(d)$, but that $c \not\leq d$ in P. Thus $L_{P(P)} \not\cong P$, and P is not a lattice.

The above examples actually contain the skeleton of an algorithm for checking posets for being lattices. We will not develop this algorithm further here, but note that it can be refined quite a bit and that some fair-sized examples, e.g., Example 7, can be handled easily using this algorithm.

<u>Remark</u>. Much of the preceding can be generalized to arbitrary lattices. The forms of the theorems vary depending on whether one wants to allow arbitrary joins or just finite joins. In the case of arbitrary joins, the generalization of Theorem 1 allows one to embed every

lattice in a complete lattice, while the generalization of Theorem 6(d) leads directly to some of Raney's results dealing with completely distributive lattices. Both theories are complicated by the fact that arbitrary lattices need not have any join-irreducible or meet-irreducible elements, and by other considerations. Actually all the above theorems hold for lattices of finite length. We have presented everything above in the context of finite lattices so that the underlying ideas would stand out more clearly. We would also like to mention that other classes of lattices (e.g., geometric lattices) can be characterized in terms of properties of their posets of irreducibles as was done in Theorem 6(d) for distributive lattices. For details see [13].

III. THE FREE DISTRIBUTIVE LATTICE ON n GENERATORS

The free distributive lattice on n generators, $FD(n)$, is $D(2^{\underline{n}})$. For basic information about $FD(n)$ the reader should consult [1; pp 34, 59] or [12]. Actually, for an arbitrary set X, $D(2^X)$ is the free completely distributive (complete) lattice on $|X|$ generators.

This contrasts with the result of H. Gaifman and A. W. Hales that there does not exist a free complete Boolean algebra with even countably many generators (see [1; p. 259]). We note that in addition to $FD(|X|)$ for infinite X, it is possible to talk about a free distributive lattice with infinitely many generators (see A. Nerode [16]).

The problem of enumerating $FD(n)$ was first proposed by Dedekind [8] in 1897. Exact answers are known with certainty only for $n \leq 6$. We now show that as is often the case, the problem of enumerating $FD(|X|)$ if X is infinite is much easier than if X is finite.

THEOREM 9. Let X be an infinite set. Then $|FD(|X|)| = |2^{2^X}|$.

<u>Proof</u>: Clearly, $FD(|X|) \leq |2^{2^X}|$. Since X is infinite, there exists a bijection $f: \underline{2} \times X \to X$. If $\gamma \in 2^X$, then we define $\gamma^* = \{f(2,\alpha) | \alpha \in \gamma\} \quad \{f(1,\alpha) | \alpha \in X - \gamma\}$. Note that $|\gamma^*| = |X|$, and that if $\gamma_1, \gamma_2 \in 2^X$, $\gamma_1 \neq \gamma_2$,

then $\gamma_1^* \not\leq \gamma_2^*$ and $\gamma_2^* \not\leq \gamma_1^*$.

Define $F: 2^{2^X} \to FD(|X|)$, by $F(S) = \{\Delta \varepsilon 2^X \mid \Delta \leq \gamma^*$ for some $\gamma \varepsilon S\}$. It is obvious from the definition that for $S \varepsilon 2^{2^X}$, $F(S)$ is closed from below, and hence F is well-defined. We claim that F is injective. Suppose that we have $F(S) = F(T)$, for $S,T \varepsilon 2^{2^X}$. Let $\lambda \varepsilon S$, then $\lambda^* \varepsilon F(S) \Rightarrow$ there exists $\gamma \varepsilon T$ such that $\lambda^* \leq \gamma^*$. But as we saw above this is only possible if $\lambda = \gamma$. Thus $\lambda \varepsilon T$ and consequently $S \leq T$. By symmetry, we get that $S \geq T$, and finally that $S = T$. Thus $|2^{2^X}| \leq |FD(|X|)$ and we are done.

From Theorem 5 we have the following results.

THEOREM 10. $FD(n)$ is irreducible and $Aut(FD(n)) \simeq S_n$ (the symmetric group on n letters).

We note that Theorem 10 is also true for $FD(n)-\{0,I\}$, which is often considered to be the free distributive lattice. This is true since $FD(n) - \{0,I\} \simeq D(2^{\underline{n}} - \{\phi,\underline{n}\})$.

It would be of interest to know the factors of $|FD(n)|$, but the irreducibility of $FD(n)$ suggests that there is no "natural" way to factor $|FD(n)|$. The only result along these lines which is known is Yamamoto's, that if n is even so is $|FD(n)|$ [17]. The converse of this statement is false, e.g., $|FD(3)| = 20$.

We conclude this paper by considering several aspects of the enumeration of $FD(n)$. We wish to briefly sketch the nature of the functions $L_k(n)$, where $L_k(n)$ is the number of elements of $FD(n)$ of cardinality k.

THEOREM 11. $L_k(n) = \sum_{\rho=\lambda_k}^{k-1} C(\rho,k)\binom{n}{\rho}$, where λ_k is an integer such that $2^{\lambda_k} \geq k > 2^{\lambda_k - 1}$ and $C(\rho,k)$ is the number of order ideals of cardinality k of $\underline{2}^{\rho}$ which contain all the singletons of $\underline{2}^{\rho}$.

<u>Remark</u>. Thus we see that, for $k \geq 1$, $L_k(n)$ is a polynomial in n of degree $k-1$, and since $C(k-1,k) = 1$, the leading coefficient is $1/(k-1)!$. $L_k(n)$ resembles the chromatic polynomial somewhat. Note that $0,1,\ldots,\lambda_k - 1$ are among the roots of $L_k(n)$. These are the only possible non-negative integral roots of $L_k(n)$, since if $n \geq \lambda_k$, there exists at least one closed from below subset of $\underline{2}^{\lambda_k}$ having cardinality k. It is possible for $L_k(n)$ to have negative integers as roots, e.g., -1 is a root of $L_4(n)$ and -9 is a root of $L_5(n)$. All the $L_k(n)$ up to $k = 7$ have only real roots each with multiplicity one. Whether this is true in general is not known to the author.

We also observe that $L_k(n) = L_{2^n - k}(n)$ for fixed k and n. Thus if we know $L_k(n)$ for $k = 1,\ldots, m$, for a given n we can calculate the elements on $2m$

levels of FD(n).

Note also that from Theorem 11, it follows that for fixed k, $L_k(n) \sim \frac{1}{(k-1)!}n^{k-1}$ as $n \to \infty$. Unfortunately, this gives some information about the tail ends of FD(n), but does not help to understand the behavior of the middle terms.

It turns out that the values of the $C(\rho,k)$'s can also be calculated from a polynomial. We will now present the machinery necessary for calculating at least some of the $L_k(n)$ fairly easily.

We should note that a somewhat similar approach to the problem of calculating FD(n) was used by Randolph Church [6], although he fixed n and let k vary. Thus in [6] he obtained the values for $L_k(n)$, $n \leq 5$ and for all k.

<u>Definition</u>. By $P(j,k)$ we shall mean $C(k-j-1,k)$, and by $C_1(a,b)$ we shall mean the number of elements of (a,b) such that no singleton is a maximal element, where (a,b) is the set of all closed from below subsets of cardinality b of $2^{\underline{a}}$ which contain all the singletons of $2^{\underline{a}}$.

<u>Remark</u>. Thus we have that $L_k(n) = \sum_{j=0}^{k-\lambda_k-1} P(j,k)\binom{n}{k-j-1}$. Note also that $P(0,k) = C(k-1,k) = 1$ for all $k \geq 0$.

The following theorem shows that for a fixed j,

$P(j,k)$ is a polynomial in k of degree $2j$.

THEOREM 12. For $j \geq 0$, $P(j,k) = \sum_{i=m_j}^{2j} C_1(i,i+j+1)\binom{k-j-1}{i}$, where m_j is the smallest integer such that $2^{m_j} \geq m_j+j+1 > 2^{m_j-1}$.

The strategies for calculating $C_1(a,b)$, $P(j,k)$, and $L_k(n)$ are involved and rather technical. The author has calculated $C_1(2j-a,3j-a+1)$ explicitly for $0 \leq a \leq 9$ and $P(j,k)$ explicitly for $0 \leq j \leq 10$. Theorem 13 gives the explicit values of $L_k(n)$ for $0 \leq k \leq 16$.

THEOREM 13. For $n \geq 0$,

(1) $L_0(n) = 1;$

(2) $L_1(n) = 1;$

(3) $L_2(n) = \binom{n}{1} = n;$

(4) $L_3(n) = \binom{n}{2} = \frac{n^2-n}{2};$

(5) $L_4(n) = \binom{n}{2} + \binom{n}{3} = \frac{n^3-n}{6};$

(6) $L_5(n) = 3\binom{n}{3} + \binom{n}{4} = \frac{n^4+6n^3-25n^2+18n}{24};$

(7) $L_6(n) = 3\binom{n}{3}+6\binom{n}{4}+\binom{n}{5} = \frac{n^5+20n^4-85n^3+100n^2-36n}{120};$

(8) $L_7(n) = \binom{n}{3} + 15\binom{n}{4} + 10\binom{n}{5} + \binom{n}{6};$

(9) $L_8(n) = \binom{n}{3} + 20\binom{n}{4} + 45\binom{n}{5} + 15\binom{n}{6} + \binom{n}{7};$

(10) $L_9(n) = 19\binom{n}{4} + 120\binom{n}{5} + 105\binom{n}{6} + 21\binom{n}{7} + \binom{n}{8};$

(11) $L_{10}(n) = 18\binom{n}{4}+220\binom{n}{5}+455\binom{n}{6}+210\binom{n}{7}+28\binom{n}{8}+\binom{n}{9};$

(12) $$L_{11}(n) = 13\binom{n}{4}+322\binom{n}{5}+1,385\binom{n}{6}+1,330\binom{n}{7}+378\binom{n}{8}+36\binom{n}{9}+\binom{n}{10};$$

(13) $$L_{12}(n) = 10\binom{n}{4}+420\binom{n}{5}+3,243\binom{n}{6}+6,020\binom{n}{7}+3,276\binom{n}{8}+630\binom{n}{9}+45\binom{n}{10}+\binom{n}{11};$$

(14) $$L_{13}(n) = 6\binom{n}{4}+500\binom{n}{5}+6,325\binom{n}{6}+21,014\binom{n}{7}+20,531\binom{n}{8}+7,140\binom{n}{9}+990\binom{n}{10}+55\binom{n}{11}+\binom{n}{12};$$

(15) $$L_{14}(n) = 4\binom{n}{4}+560\binom{n}{5}+10,925\binom{n}{6}+59,619\binom{n}{7}+99,680\binom{n}{8}+58,989\binom{n}{9}+14,190\binom{n}{10}+1,485\binom{n}{11}+66\binom{n}{12}+\binom{n}{13};$$

(16) $$L_{15}(n) = \binom{n}{4}+600\binom{n}{5}+17,345\binom{n}{6}+145,050\binom{n}{7}+393,540\binom{n}{8}+379,848\binom{n}{9}+149,115\binom{n}{10}+26,235\binom{n}{11}+2,145\binom{n}{12}+78\binom{n}{13}+\binom{n}{14};$$

(17) $$L_{16}(n) = \binom{n}{4}+616\binom{n}{5}+25,945\binom{n}{6}+314,965\binom{n}{7}+1,313,260\binom{n}{8}+1,992,144\binom{n}{9}+1,226,919\binom{n}{10}+341,220\binom{n}{11}+45,760\binom{n}{12}+3,003\binom{n}{13}+91\binom{n}{14}+\binom{n}{15}.$$

Remark. Note that we have enough information to calculate $L_{17}(n)$ entirely, since we know from Theorem 11 that $C(5,17) = L_{17}(5)$ and from the Remark following Theorem 11 that $L_{17}(5) = L_{15}(5)$. All the remaining coefficients can be calculated using the values of $P(j,k)$ for $0 \leq j \leq 10$.

The ideas in this chapter have been applied by Butler and the author [5] to the enumeration of partially ordered sets, to show that when partially ordered sets are broken down into certain classes, each class is enumerated by a polynomial.

We conclude by briefly discussing the problem of finding an accurate upper bound for $|FD(n)|$. The best published result is that of D. J. Kleitman [12] which states that $|FD(n)| \leq 2^{(1+k\, n^{-\frac{1}{2}} \ln n)} E_n$ for some constant k, where $E_n = \binom{n}{[\frac{n}{2}]}$. Recently, Kleitman and the author working jointly have been able to improve this upper bound. In particular, we have shown that $|FD(n)| \leq 2^{(1+k\, n^{-1} \ln n)} E_n$. The improvement of the upper bound follows from a detailed analysis of Hansel's approach to the problem [10], using a characterization, due to Greene and Kleitman [9], of the partition of $\underline{2^n}$ into chains used by Hansel. Greene and Kleitman characterize this partition in terms of the way an expression can be parenthesized allowing a certain number of "free" parentheses to remain.

It can be shown that [13] $|FD(n)| \geq 2^{(1 + c 2^{-[\frac{n}{2}]})} E_n$, for c a constant on the order of 1 and appropriate n.

The lower bound given in [12] is too large to be supported by the argument given there.

We wish to finish by stating two conjectures. The first is that the order of FD(n) is closer to the lower bound given above than it is to the upper bound given above. The second conjecture concerns the number of anti-chains of $2^{\underline{n}}$ (recall that anti-chains of $2^{\underline{n}}$ correspond in a 1-1 fashion to the sets of maximal elements of elements of $D(2^{\underline{n}})$). This conjecture is due to Garrett Birkhoff and asserts that asymptotically all anti-chains of $2^{\underline{n}}$ consist entirely of subsets of $\underline{n}$ with cardinality between $[\frac{n}{2}]-k$ and $[\frac{n}{2}]+k$, where k is a small fixed integer, perhaps 3, 4 or 5.

The author would like to gratefully acknowledge many stimulating discussions with Garrett Birkhoff, Daniel J. Kleitman, and Curtis Greene, and the many very helpful suggestions which they made.

References

[1] Birkhoff, G., Lattice Theory. 3rd ed. AMS Colloq. Publ. Vol. XXV, Providence, R.I., 1967.

[2] Birkhoff, G. and O. Frink, Representations of Lattices by Sets, Trans. AMS 64, 299-316.

[3] Brandon, R. L., K.K.-H. Butler, D. W. Hardy and G. Markowsky, Cardinalities of D-classes in B_n. Semigroup Forum 4(1972),341-344.

[4] Brandon, R. L., D. W. Hardy and G. Markowsky, The Schutzenberger Group of an H-Class in the Semigroup of Binary Relations, Semigroup Forum 5 (1972), 45-53.

[5] Butler, K.K.-H. and G. Markowsky, The Number of Partially Ordered Sets II. Submitted to the Journal of Combinatorial Theory.

[6] Church, R., Numerical Analysis of Certain Free Distributive Structures. Duke Math. Journal 6 (1940) 732-734.

[7] Crapo, H. H. and G.-C. Rota, On the Foundations of Combinatorial Theory: Combinatorial Geometries. (preliminary edition) M.I.T. Press, Cambridge, Mass., 1970.

[8] Dedekind, R. Uber Zerlegunzen von Zablen durch ihre grossten gemeinsamen Teiler. Festschrift Hoch. Braunschweig u. ges. Werke, II (1897), 103-148.

[9] Greene, C. and D. J. Kleitman, Strong Versions of Sperner's Theorem. To appear in Advances in Math.

[10] Hansel, G., Sur le nombre des fonctions booleennes monotones de n variables. C. R. Acad. Sci. Paris 262 (1966), 1088-1090

[11] Hillman, A., On the Number of Realizations of a Hasse Diagram by Finite Sets. Proc. AMS 6 (1955) 542-8.

[12] Kleitman, D., On Dedekind's Problem: The Number of Monotone Boolean Functions. Proc. AMS 21 (1969) 677-682.

References (Cont.)

[13] Markowsky, G., Combinatorial Aspects of Lattice Theory With Applications to the Enumeration of Free Distributive Lattices, Ph.D. Thesis, Harvard University, 1973.

[14] Markowsky, G., Green's Equivalence Relations and the Semigroup of Binary Relations on a Set. 1970. Privately circulated, 146 pp.

[15] Markowsky, G., Idempotents and Product Representations with Applications to the Semigroup of Binary Relations, Semigroup Forum 5 (1972), 95-119.

[16] Nerode, A., Composita, Equations, and Freely Generated Algebras. Trans. AMS 91 (1959) 139-51.

[17] Yamamoto, K., Note on the Order of Free Distributive Lattices. The Science Reports of the Kanazawa University, Vol. II, No. 1, March, 1953, pp. 5-6.

[18] Zaretskii, K. A., The Representations of Lattices by Sets. Uspekhi Mat. Nauk (Russian) 16 (1961) 153-154.