Reprinted from Journal of Mathematical Analysis and Applications Vol. 63, No. 1, March 15, 1978
All Rights Reserved by Academic Press, New York and London *Printed in Belgium*

Differential Operators and the Theory of Binomial Enumeration

George Markowsky

Computer Science Department, IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598

Submitted by G.-C. Rota

1. Introduction

In [4], Rota and Mullin established a very interesting correspondence between polynomials of "binomial type" and a family of linear operators they call "delta operators." In this paper we consider the family of all differential operators (linear operators which lower degree by one) on the ring of polynomials over a field, F, of characteristic 0. We show that this family breaks up into subfamilies based on commutativity.

We show many of the results in [4], generalize to these subfamilies, and that these results depend upon straightforward manipulations of formal power series and the use of Taylor's Theorem, rather than on specific substitution-type arguments. These methods allow us to show that in order for an operator to be shift-invariant (i.e., to commute with all shift-operators) it is enough for it to commute with a single *nontrivial* ($\neq I$) shift operator. The key result here is that given two differential operators G and H, G can be expanded in terms of H if and only if $GH = HG$.

We derive the operator form of the Pincherle Derivative (the formal power series derivative) and use this result to extend the closed form formulas of Rota and Mullin to the "basic families" of polynomials of arbitrary differential operators. Our formulas allow one to find "basic families" in terms of any other convenient basic family, thus removing the asymmetry of Rota–Mullin's work.

Furthermore, we show that for any differential operator G and any sequence $a, c_1, c_2, \ldots,$ of elements of F, with $a, c_1 \neq 0$, there exists a unique differential operator H, commuting with G, such that the basic family $(Q_0, Q_1, \ldots)$ of H satisfies $Q_i(a) = c_i$ for all $i \geqslant 1$.

We also exhibit a simple relationship between the various maximal commuting families of differential operators which also relates the various basic families to each other.

Finally, we show that given any maximal commuting family of differential operators and their associated basic families, there exists a unique family of linear operators L_y such that a differential operator G is in the commuting family

0022-247X/78/0631 0145$02.00/0

if and only if there exists $y \neq 0$, such that for all $n \geqslant 1$, $L_y Q_n(x) = \sum_{i=0}^{n} \binom{n}{i} Q_i(x) Q_{n-i}(y)$, where $\{Q_0, Q_1, \ldots\}$ is the basic family of G. Note that polynomials of binomial type are just the case where $L_y = E^y$ (where $E_y P(x) = P(x+y)$ for all polynomials P).

Actually, this proves the somewhat stronger result, that in order to show that a normal family of polynomials is of binomial type we need only find one value $c \neq 0$ such that for all n, $Q_n(x+c) = \sum_{i=0}^{n} \binom{n}{i} Q_i(x) Q_{n-i}(c)$.

We also show that the family of delta operators is characterized by the property that $L_y L_z = L_{y+z}$.

We feel that our presentation of these results is particularly simple in terms of the amount of machinery we require. Almost everything is seen to depend on commutativity of operators and simple manipulations of formal power series. We note that a number of additional results and approaches can be found in [2, 3, and 6].

2. Fundamentals

Throughout, we shall be concerned with the ring of all polynomials, $F[x]$, over some field F of characteristic 0.

Definition 2.1. A *differential operator* L: $F[x] \to F[x]$ is a linear operator such that:

(a) $L(f) = 0$ if and only if $f \in F$;

(b) $\deg(L(f)) = \deg(f) - 1$ if $f \notin F$.

Remark. Clearly, if $Lf = Lg$, then $f - g \in F$.

Definition 2.2. A *normal family* of polynomials, $\{P_0, P_1, \ldots\}$, in $F[x]$ has the following properties:

(a) P_i is of degree i;

(b) $P_0 \equiv 1$;

(c) $P_i(0) = 0$ for all $i \geqslant 1$.

Remark. A normal family forms a basis for $F[x]$ over F.

Definition 2.3. Given a differential operator L, we say that a family of polynomials $\{Q_0, Q_1, \ldots\}$ is a *basic family for* L if:

(a) $\{Q_0, Q_1, \ldots\}$ is a normal family;

(b) $LQ_{i+1} = (i+1) Q_i$.

The following theorem exhibits the interrelations between the preceding definitions.

Theorem 2.4. (a) *Given a differential operator L, there exists a* unique *basic family for L.*

(b) *Given a normal family $N = \{P_0, P_1, \ldots\}$, there exists a* unique *differential operator L, such that N is a basic family for L.*

(c) *Let $N = \{P_0, P_1, \ldots\}$ be the basic family for the differential operator L. Then for all $f \in F[x]$, $f = f(0) + [(Lf)(0)/1!]\, P_1 + \cdots + [(L^n f)(0)/n!]\, P_n$, where $n = \deg f$.*

Proof. The proofs are all straightforward and may be found in Berge [1, pp. 73–74]. ∎

We assume that the reader knows enough about formal power series to make the following definition precise. A basic reference on formal power series is Niven [5].

Definition 2.5. Let L be a differential operator. By Form $[L]$ we mean the ring of all *formal power series in* L. If $G = a_0 I + a_1 L + \cdots \in$ Form $[L]$, we use G_i to denote a_i.

Remark. Every element of Form $[L]$ has a natural interpretation as a linear operator on $F[x]$. Let $G \in$ Form$[L]$, then we define $Gf = \sum_{i=0}^{\infty} G_i(L^i f)$. This sum is well-defined since $L^i f = 0$ for all $i > \deg f$.

The next theorem shows that every element $G \in$ Form$[L]$, has a unique power series expansion. Furthermore, it is easy to show that if $G, H \in$ Form$[L]$, then the composition GH is in Form$[L]$, and its expansion is given by the product of the expansions of G and H. Similarly, the power series expansion of $G + H$ is simply the formal sum of the expansions of G and H.

Theorem 2.6. *Let L be a differential operator and $G, H \in$ Form$[L]$. If $G(f) = H(f)$ for all $f \in F[x]$, then $G_i = H_i$ for all $i \geqslant 0$.*

Proof. Suppose $G_i \neq H_i$ for some i. Let i_0 be such that $G_i = H_i$ for all $i < i_0$ and $G_{i_0} \neq H_{i_0}$. Let $\{P_0, P_1, \ldots\}$ be the basic family for L (by Theorem 2.4). However, $(G - H) P_{i_0} = (G_{i_0} - H_{i_0})\, i_0! \neq 0$, contradicting the fact that $GP_{i_0} = HP_{i_0}$. ∎

3. The Main Result

In this section we derive a very simple characterization of those linear operators which are in Form$[L]$ for a given differential operator L. This result allows us to achieve our other results in a simple and straightforward manner using power series expansions. We note that part (a) is essentially the content of 2.3 in [2], while part (b) is Rota–Mullin's version of Taylor's Theorem.

THEOREM 3.1. *Let L be a differential operator and G be any linear operator from $F[x]$ to $F[x]$. Then*

(a) *$G \in \text{Form}[L]$ if and only if $GL = LG$;*

(b) *if $G \in \text{Form}[L]$, then $G_i = (GP_i)(0)/i!$, where $\{P_0, P_1, \ldots\}$ is the basic family for L.*

Proof. As we noted in the remark following Definition 2.5, in Form[L], composition of operators corresponds to multiplication of power series. Thus composition is commutative.

Conversely, if $GL = LG$, we will show that $G = \sum_{i=0}^{\infty} [(GP_i)(0)/i!] L^i$. Thus (b) follows.

Let $H = \sum_{i=0}^{\infty} [(GP_i)(0)/i!] L^i$. We will show that $GP_i = HP_i$ for all i. Since, $\{P_0, P_1, \ldots\}$ is a basis for $F[x]$, it then follows that $G = H$.

We first note that for all $f \in F[x]$, $\deg Gf \leqslant \deg f$. Let $\deg f = k$, then $L^{k+1}f = 0$. Since $LG = GL$, $0 = GL^{k+1}f = L^{k+1}Gf$, which can happen if and only if $\deg Gf \leqslant k$. Thus G applied to a constant yields a constant. In particular, we have $HP_0 = H1 = (GP_0)(0) = GP_0$, since GP_0 is a constant.

We now show that for all i, $HP_i = GP_i$. By substitution, $HP_i = \sum_{j=0}^{i} \binom{i}{j} GP_j(0) P_{i-j}$. Since $\deg GP_i \leqslant \deg P_i$. there exist unique $b_0, b_1, \ldots, b_i \in F$ such that $GP_i = \sum_{j=0}^{i} b_i P_{i-j}$. We will now calculate the b_j's:

$$(1) \quad (L^{i-j}GP_i)(0) = (GL^{i-j}P_i)(0) = (GL^{i-j}P_i)(0) = (Gi^{(i-j)}P_j)(0) = i^{(i-j)}(GP_i)(0);$$

$$(2) \quad (L^{i-j}GP_i)(0) = \left(L^{i-j}\left(\sum_{j=0}^{i} b_j P_{i-j}\right)\right)(0) = \left(\sum_{k=0}^{i} b_{j+k}(i-j+k)^{(i-j)} P_k\right)(0) = b_j(i-j)!,$$

since $P_k(0) = 0$ for all $k \geqslant 1$ and $P_0(0) = 1$. From (1) and (2) we get that $b_j = \binom{i}{j} GP_j(0)$. Thus $G = H$. ∎

THEOREM 3.2. *Let L be a differential operator and $H \in \text{Form}[L]$. Then H is a differential operator if and only if $H_0 = 0$ and $H_1 \neq 0$.*

Proof. If H is a differential operator, $H(\text{const}) = 0$, which implies $H_0 = 0$. Since $Hx \neq 0$, we have that $H_1 \neq 0$.

Conversely, if $H_0 = 0$, $H(\text{const}) = 0$. Furthermore, if $H_1 \neq 0$, by standard results on formal power series, there exists $K \in \text{Form}[L]$ such that $HK = KH = L$. If $Hf = 0$ for some $f \in F[x]$, we have $0 = KHf = Lf$, which implies that $f \in F$. Thus $Hf = 0$ if and only if $f \in F$.

We noted in the proof of Theorem 3.1, that for all $G \in \text{Form}[L]$ and $f \in F[x]$, $\deg Gf \leqslant \deg f$. Thus for $f \in F[x] - F$, we have $\deg f > \deg Hf \geqslant \deg KHf = \deg Lf = \deg f - 1$, i.e., $\deg Hf = \deg f - 1$. ∎

The following lemma is very useful for establishing the membership of a given operator in some Form[L].

LEMMA 3.3. *Let L be a differential operator and G a linear operator such that $GL = LG$ and for some $c_0 \in F$, $G - c_0 I$ is a differential operator (where I is the identity operator on $F[x]$). Let H be a linear operator such that $GH = HG$. Then $HL = LH$, i.e., $H \in \text{Form}[L]$.*

Proof. Let $T = G - c_0 I$ be the differential operator alluded to above. Clearly, $HT = TH$, and by Theorem 3.1, $H \in \text{Form}[T]$. Also it is clear that $LT = TL$, i.e., $L \in \text{Form}[T]$. Since Form[T] is commutative, $LH = HL$. ∎

Remark. The above results enable us to derive many of the results in sections 3 and 4 of Rota–Mullin [4].

DEFINITION 3.4. (a) For any $a \in F$, let $E^a: F[x] \to F[x]$ be given by $E^a f(x) = f(x + a)$. The E^a are called *shift-operators*. We will use E to denote E^1.

(b) Let $\Delta: F[x] \to F[x]$ be given by $\Delta f(x) = f(x+1) - f(x)$. Δ is called the difference operator. Clearly, Δ is a differential operator.

(c) We shall define an operator G to be *shift-invariant* if $EG = GE$. G will be said to be a *delta-operator* if G is shift-invariant and Gx is a *nonzero* constant.

(d) A family of polynomials $\{P_0, P_1, \ldots\}$ is said to be of binomial type if it is a normal family and for all $n \geqslant 1$, x and y $P_n(x+y) = \sum_{i=0}^{n} \binom{n}{i} P_i(x) P_{n-1}(y)$.

Remark. Rota–Mullin define shift-invariant to mean $E^a G = GE^a$ for all a. As we shall see below, their definition turns out to be equivalent to our definition above. This fact is also noted by Garsia [3, Remark 1.1].

THEOREM 3.5. (1) *An operator G is shift-invariant if and only if $G \in \text{Form}[\Delta]$.*

(2) *An operator G is shift-invariant if and only if $GE^a = E^a G$ for all a.*

(3) *G is a delta operator if and only if G is a shift-invariant differential operator.*

(4) *If G and H are delta operators, either one can be expanded as an infinite power series in terms of the other.*

Proof. (1) Since Δ is a differential operator and $E - I = \Delta$, we need only apply Lemma 3.3 to see that $G \in \text{Form}[\Delta]$. The converse follows by commutativity of Form [Δ].

(2) Since, $E^a E = EE^a$ for all a, $E^a \in \text{Form}[\Delta]$. Since Form[$\Delta$] is commutative, $GE^a = E^a G$ since by (1), $G \in \text{Form}[\Delta]$.

(3) If G is a delta-operator, $G \in \text{Form}[\Delta]$, i.e., $G = G_0 I + G_1 \Delta + \cdots$. But $Gx = G_0 x + G_1 =$ nonzero constant. Hence $G_0 = 0$ and $G_1 \neq 0$. Thus by Theorem 3.2, H is a differential operator. The converse is straightforward.

(4) Since G and H are both in Form$[\Delta]$, $GH = HG$ and the result follows from Theorem 3.1. ∎

4. The Pincherle Derivative and Closed Formulas for Basic Families

Definition 4.1. Let L be a differential operator and $G \in \text{Form}[L]$ have the power series expansion $G_0 I + G_1 L + \cdots$. We define (following Rota–Mullin [4]) the *L-Pincherle Derivative* $\mathscr{D}_L(G)$ of G to be the formal derivative of L in Form $[L]$, i.e., $\mathscr{D}_L(G)$ is the element of Form $[L]$ with the power series expansion $G_1 I + 2G_2 L + 3G_3 L^2 + \cdots$. Note $\mathscr{D}_L(A + B) = \mathscr{D}_L(A) + \mathscr{D}_L(B)$ and $\mathscr{D}_L(AB) = A\mathscr{D}_L(B) + B\mathscr{D}_L(A)$, etc

Remark. The next proposition gives the operator form of the Pincherle Derivative. It generalizes Proposition 1 of [4, p. 192]. Fillmore and Williamson derive a similar result [2, Proposition 2.8].

Theorem 4.2. *Let L, G and $\mathscr{D}_L(G)$ be as in Definition 4.1. Let $P_0, P_1, \ldots,$ be the basic family associated with L and $M: F[x] \to F[x]$ the linear operator given by $MP_i = P_{i+1}$ for all i. Then $\mathscr{D}_L(G) = GM - MG$.*

Proof. We will show that $\mathscr{D}_L(G) P_i = (GM - MG) P_i$ for all i. Since $P_0, P_1, \ldots,$ is a basis for $F[x]$ it will follow that $\mathscr{D}_L(G) = GM - MG$.

(a) $$\mathscr{D}_L(G) P_i = \sum_{j=0}^{i} (j+1) G_{j+1} i^{(j)} P_{i-j}$$

(where $a^{(b)} = (a)(a-1) \cdots (a-b+1)$ and $a^{(0)} = 1$)

$$= \sum_{j=1}^{i+1} j G_j i^{(j-1)} P_{i+1-j}.$$

(b) $$(GM - MG) P_i = GP_{i+1} - MGP_i$$

$$= \sum_{j=0}^{i+1} G_j (i+1)^{(j)} P_{i+1-j} - M\left(\sum_{j=0}^{i} G_j i^{(j)} P_{i-j}\right)$$

$$= \sum_{j=0}^{i+1} G_j ((i+1)^{(j)} - i^{(j)}) P_{i+1-j}.$$

However, for $j \geq 1$, $(i+1)^{(j)} - i^{(j)} = (i+1) i^{(j-1)} - (i-j+1) i^{(j-1)} = j i^{(j-1)}$. Thus $\mathscr{D}_L(G) = GM - MG$. ∎

We can now duplicate Rota–Mullin's [4] calculation of closed forms for the basic family of a differential operator.

Theorem 4.3. *Let L and G be differential operators such that $LG = GL$, i.e., $G \in \text{Form}[L]$. Let $P_0, P_1, \ldots,$ be the basic family for L and $Q_0, Q_1, \ldots,$ the basic family for G. Then, for all $n \geq 1$,*

(a) $Q_n = \mathscr{D}_L(G) B^{-n-1} P_n = B^{-n} P_n - \mathscr{D}_L(B^{-n}) P_{n-1} = MB^{-n} P_{n-1}$; *and*
(b) $Q_n = M\mathscr{D}_L(G)^{-1} Q_{n-1}$.

Here B is the unique linear operator such that $G = LB$ and $M: F[x] \to F[x]$ is the linear operator used in Theorem 4.2 ($MP_i = P_{i+1}$ for all $i \geq 0$).

Proof. (a) Observe that for all $n \geq 1$,

$$\begin{aligned} \mathscr{D}_L(G) B^{-n-1} P_n &= \mathscr{D}_L(LB) B^{-n-1} P_n \\ &= \mathscr{D}_L(L) B^{-n} P_n + \mathscr{D}_L(B) B^{-n-1} LP = B^{-n} P_n + n\mathscr{D}_L(B) B^{-n-1} P_{n-1} \\ &= B^{-n} P_n - \mathscr{D}_L(B^{-n}) P_{n-1} = B^{-n} P_n - (B^{-n} M - MB^{-n}) P_{n-1} \\ &= B^{-n} P_n - B^{-n} MP_{n-1} + MB^{-n} P_{n-1} = MB^{-n} P_{n-1}. \end{aligned}$$

Let $H_0 = 1$ and for $n \geq 1$ $H_n = \mathscr{D}_L(G) B^{-n-1} P_n$. Then clearly, for $n \geq 2$ $GH_n = LBH_n = n\mathscr{D}_L(G) B^{-n} P_{n-1} = nH_{n-1}$. Finally, $GH_1 = \mathscr{D}_L(G) B^{-1} P_0 = P_0 = H_0$, since if $G = G_1 L + G_2 L^2 + \cdots$, $B^{-1} = (1/G_1) I + \cdots$, and $\mathscr{D}_L(G) = G_1 I + \cdots$. Since $H_n = MB^{-n} P_{n-1}$ for $n \geq 1$, $H_n(0) = 0$.

Note, $MT(0) = 0$ for any $T \in F[x]$, because MT can be written as a linear sum of $P_1, P_2, P_3, \ldots$. By the uniqueness of the basic family, $H_n = Q_n$.

(b) $Q_1 = MB^{-1} P_0 = M\mathscr{D}_L(G)^{-1} P_0 = M\mathscr{D}_L(G)^{-1} Q_0$, since $\mathscr{D}_L(G) = G_1 I + \cdots$ and $B = G_1 I + \cdots$. For $n \geq 2$, $Q_n = MB^{-n} P_{n-1}$ and $Q_{n-1} = \mathscr{D}_L(G) B^{-n} P_{n-1}$. Thus $P_{n-1} = \mathscr{D}_L(G)^{-1} B^n Q_{n-1}$, whence $Q_n = M\mathscr{D}_L(G)^{-1} Q_{n-1}$. ∎

The next theorem shows that the basic families associated with a maximal family can separate sequences of elements of F.

Theorem 4.4. *Let G be a differential operator and $a, c_1, c_2, \ldots,$ elements of F with $a, c_1 \neq 0$. There exists a unique $H \in$ Form $[G]$ such that the basic family $Q_0, Q_1, \ldots,$ of H satisfies $Q_i(a) = c_i$ for all $i \geq 1$. (Note we require $a \neq 0$ because $Q_i(0) = 0$ for $i \geq 1$ and $c_1 \neq 0$ since $Q_1(t) \neq 0$ if $t \neq 0$).*

Proof. Let $P_0, P_1, \ldots,$ be the basic family for G. Any differential operator, H, in Form $[G]$ has an expansion $b_1 G + b_2 G^2 + \cdots$, with $b_1 \neq 0$. We shall show how to use Theorem 4.3 and the requirement $Q_i(a) = c_i$ for $i \geq 1$ to calculate b_i.

From Theorem 4.3, $Q_n = MB^{-n}P_{n-1}$ for $n \geqslant 1$ where $MP_i = P_{i+1}$ for all i and $B = b_1I + b_2G + \cdots$, i.e., $H = GB$. Now $B^{-1} = e_1I + e_2G + e_3G^2 +$ where the e_i are functions of the b_i. In particular, $e_1 = 1/b_1$. Note that B and B^{-1} uniquely determine one another. We shall actually determine the e_i uniquely, from which it will follow that the b_i are unique. More generally, $B^{-n} = \sum_{i=0}^{\infty} R(n,i)\, G^i$ with $R(n,i) = ne_1^{n-1}e_{i+1} + T(n,i) = (n/b_1^{n-1})\, e_{i+1} + T(n,i)$ where $T(n,i)$ is a polynomial in $e_1,\ldots, e_i$. Thus $Q_n = M(\sum_{i=0}^{n-1} \lambda_i R(n, n-1-i)\, P_i) = \sum_{i=0}^{n-1} \lambda_i R(n, n-1-i)\, P_{i+1}$ where $\lambda_i = (n-1)(n-2)\cdots(n-i)$. We have $(*)\ c_n = Q_n(a) = (n-1)!\, R(n, n-1)\, P_1(a) + \sum_{i=1}^{n-1} \lambda_i\, R(n, n-1-i)\, P_{i+1} = n!\, e_1^{n-1}e_nP_1(a) + W(n,a)$ where $W(n,a)$ is a polynomial in $e_1,\ldots, e_{n-1}$. Since $P_1(x) = dx$ for some nonzero $d \in F$, for $n \geqslant 1$ we can solve (*) to get $e_n = (c_n - W(n,a))/n!\, e_1^{n-1}$. For $n = 1$, we have $c_1 = Q_1(a) = MB^{-1}P_0\,|_{x=a} = P_1(a)$. Thus $e_1 = c_1/P_1(a) = 0$. Thus we can solve (*) inductively to find B^{-1}, B and finally H. ∎

We conclude this section with the following theorem which establishes the relationship between two arbitrary differential operators and their associated maximal commuting familes.

THEOREM 4.5. *Let G and H be differential operators, and $P_0, P_1,\ldots,$ and $Q_0, Q_1,\ldots,$ their respective basic families. Let $L\colon F[x] \to F[x]$ be the unique linear transformation such that $LP_i = Q_i$ for all i. Then the following are true.*

(a) *L is invertible.*

(b) *The maps θ: Form $[G] \to$ Form $[H]$ and λ: Form $[H] \to$ Form $[G]$ given by $\theta(G') = LG'L^{-1}$ and $\lambda\colon (H') = L^{-1}H'L$ are isomorphisms between the two power series rings.*

(c) *If $R_0, R_1 \ldots (S_0, S_1 \ldots)$ is the basic family of a differential operator $G' \in$ Form $[G]$ ($H' \in$ Form $[H]$) then $LR_0, LR_1, \ldots$ ($L^{-1}S_0, L^{-1}S_1, \ldots$) is the basic family of the differential operator $\theta(G')(\lambda(H'))$.*

Proof. (a) Trivial since L^{-1} is defined by $L^{-1}Q_i = P_i$ for all i.

(b) We first show that $\theta(G) = H$. Note that for all $i \geqslant 1$, $(LGL^{-1})\, Q_i = (LGL^{-1})\, LP_i = LGP_i = iLP_{i-1} = iQ_{i-1}$ and $LGL^{-1}Q_0 = LGP_0 = 0$. Thus since $\theta(G)$ and H agree on a basis of $F[x]$ they are equal. For $G' \in$ Form $[G]$, $\theta(G')\, H = (LG'L^{-1})(LGL^{-1}) = LG'GL^{-1} = LGG'L^{-1} = (LGL^{-1})(LG'L^{-1}) = H\theta(G')$. Thus $\theta(G') \in$ Form $[H]$. We have actually shown that θ preserves multiplication and clearly it preserves addition. Note that the same proofs hold for λ. However, it is clear that λ and θ are inverses and hence they are both isomorphisms.

(c) Note that $G' \in$ Form $[G]$ is a differential operator if and only if there exists a $G^* \in$ Form $[G]$ such that $G'G^* = G$. Thus $H = \theta(G) = \theta(G')\, \theta(G^*)$, whence $\theta(G')$ is a differential operator in Form $[H]$. Note that $R_0 = P_0 \equiv 1$ so that $LR_0 = R_0 \equiv 1$. Since $LP_i = Q_i$, L preserves degree and for all $f \in F[x]$ $f(0) = 0$ implies $(Lf)(0) = 0$ ($f(0) = 0$ implies that $f = a_1P_1 + a_2P_2 + \cdots + a_nP_n$ and $Lf = a_1Q_1 + \cdots + a_nQ_n$. But $Q_i(0) = 0$ for $i \geqslant 1$). Thus $LR_0, LR_1,\ldots,$ is a normal family. As in (b), for $i \geqslant 1$, $\theta(G')\, LR_i = LG'R_{i-1}$ and $\theta(G')\, LR_0 = 0$. Thus $LR_0, LR_1,\ldots,$ is the basic family for $\theta(G')$. ∎

5. The Question of Identities

Clearly, a normal family of polynomials $\{P_0, P_1,\ldots\}$, is of binomial type if for all $n \geqslant 1$, $(*) E^yP_n(x) = \sum_{i=0}^{n} \binom{n}{i} P_i(x)\, P_{n-i}(y)$. We cannot expect basic families of arbitrary differential operators to satisfy neat identities involving the shift-operator, since in general they do not commute with the shift-operator. However, we have seen that commutativity has been the key property in achieving our results.

However, all is not lost, for we will show that for each differential operator L, there exists a family of linear operators L_y, such that for all differential operators G, $G \in$ Form$[L]$ if and only if for all $n \geqslant 1$, $L_yP_n(x) = \sum_{i=0}^{n} \binom{n}{i} P_i(x)\, P_{n-i}(y)$, where $\{P_0, P_1,\ldots\}$ is the basic family of G. This generalizes Theorem 1 of [4], which establishes the one–one correspondence between delta operators and polynomials of binomial type. In this scheme of things, $E^y = \Delta_y$. Furthermore, it follows that L is a delta-operator if and only if for all y, z $L_{y+z} = L_yL_z$.

DEFINITION 5.1. Let L be a differential operator and $\{P_0, P_1 \ldots\}$ its basic family. For each $y \in F$ (or a variable $y \neq x$), we define a linear transformation $L_y\colon F[x] \to F[x]$ by $L_yP_n(x) = \sum_{i=0}^{n} \binom{n}{i} P_i(x)\, P_{n-i}(y)$.

Remark. Throughout this section, L, L_y and $\{P_0, P_1,\ldots\}$ will have the meanings ascribed to them above.

LEMMA 5.2. *For all $y \in F$, $L_y \in$ Form$[L]$. Furthermore, for all $T \in F[x]$, $[L_yT(x)]_{x=0} = T(y)$. Note that $L_y = \sum_{n=0}^{\infty} [P_n(y)/n!]\, L^n$. (Recall also that $P_1(y) = 0$ if and only if $y = 0$).*

Proof.

$$LL_yP_n(x) = \sum_{i=0}^{n} \binom{n}{i} LP_i(x)\, P_{n-i}(y) \qquad (y \text{ is treated as a constant})$$

$$= \sum_{i=0}^{n} \binom{n}{i} iP_{i-1}(x)\, P_{n-i}(y) = n \sum_{i=0}^{n-1} \binom{n-1}{i} P_i(x)\, P_{n-1-i}(y)$$

$$= nL_yP_{n-1}(x) = L_yLP_n(x).$$

Thus by Theorem 3.1, $L_y \in$ Form$[L]$. $[L_yP_n(x)]_{x=0} = P_n(y)$ for all n. The general result now follows by linearity and the fact that $P_0, P_1,\ldots,$ is a basis for $F[x]$. ∎

THEOREM 5.3. *Let G be a differential operator and $Q_0, Q_1, \ldots,$ be the basic family associated with G. Then for some $y \neq 0$, and all n $L_y Q_n(x) = \sum_{i=0}^{n} \binom{n}{i} Q_i(x) Q_{n-i}(y)$* if and only if $G \in \text{Form}[L]$. *Thus* $G \in \text{Form}[L]$ *if and only if $G_y = L_y$ for some $y \neq 0$ (and hence $G_y = L_y$ for all y).*

Proof. Necessity. Arguing as in Lemma 5.2, we see that $GL_y = L_y G$ for all y. By Lemma 5.2, if $y \neq 0$ we can apply Lemma 3.3 to conclude that $G \in \text{Form}[L]$.

Sufficiency. If $G \in \text{Form}[L]$, $GL_y = L_y G$. Thus using Taylor series to calculate $L_y Q_n(x)$ (Theorem 2.4(c)), we get

$$L_y Q_n(x) = \sum_{i=0}^{n} \frac{[G^i L_y Q_n(x)]_{x=0}}{i!} Q_i(x) = \sum_{i=0}^{n} \frac{[L_y n^{(i)} Q_{n-i}(x)]_{x=0}}{i!} Q_i(x)$$

$$= \sum_{i=0}^{n} \binom{n}{i} Q_{n-i}(y) Q_i(x)$$

since $[L_y T(x)]_{x=0} = T(y)$ by Lemma 5.2. ∎

Remark. Observe that if L is any delta operator, $L_y = E^y$. But $E^y Q_n(x) = Q_n(x+y)$. Thus Theorem 5.3 yields the fact that a normal family is a basic sequence for a delta operator if and only if it is of binomial type.

Actually, we have derived a stronger result.

COROLLARY 5.4. *Let $N = \{Q_0, Q_1, \ldots\}$ be a normal family. Then N is of binomial type if and only if there exists some $c \in F - \{0\}$ such that for all n*

$$Q_n(x+c) = \sum_{i=0}^{n} \binom{n}{i} Q_i(x) Q_{n-i}(c).$$

Proof. Necessity is trivial. Sufficiency follows from Theorem 5.3 and the fact that N has some differential operator associated with it (Theorem 2.4(b)). ∎

The following corollary distinguishes the polynomials of binomial type in terms of a basic property of L_y.

COROLLARY 5.5. $L_y \in \text{Form}[\Delta]$ *if and only if for all $y, z \in F$, $L_y L_z = L_{y+z}$.*

Proof. If $L_y \in \text{Form}[\Delta]$, we noted in the remark above that $L_y = E^y$. Thus $L_y L_z = E^y E^z = E^{y+z} = L_{y+z}$. Conversely, by Lemma 5.2, $L_y = \sum_{j=0}^{\infty} [P_j(y)/j!] L^j$. Thus $L_y L_z = L_{y+z}$ implies that

$$\sum_{j=0}^{\infty} \left(\sum_{i=0}^{j} \frac{P_i(y) P_{j-i}(z)}{i!\,(j-i)!} \right) L^j = \sum_{j=0}^{\infty} \frac{P_j(y+z)}{j!} L^j.$$

This implies that $P_j(y+z) = \sum_{i=0}^{j} \binom{j}{i} P_i(y) P_{j-i}(Z)$, i.e., the P's are of binomial type. Thus $L \in \text{Form}[\Delta]$ and thus $L_y \in \text{Form}[\Delta]$. ∎

LEMMA 5.6. *Let A be a differential operator. Let $r_1, r_2, \ldots$ ($r_1 \neq 0$) be an arbitrary sequence in F. Then there exists a unique differential operator $B \in \text{Form}[A]$ such that $A = r_1 B + r_2 B^2 + \cdots$.*

Proof. The proof is essentially a straightforward computation. If we set $B = x_1 A + x_2 A^2 + \cdots$ and substitute into the above relation we must solve the following system of equations: $r_1 x_1 = 1$, $r_1 x_1 + r_2 x_1^2 = 0, \ldots, r_1 x_n + \tilde{P}_n(x_1, \ldots, x_{n-1}) = 0, \ldots$ (where $\tilde{P}_n$ is some polynomial). Clearly this system can be solved recursively for the x_i's, since $r_1 \neq 0$. ∎

The following Theorem generalizes Corollary 1 of [2, p. 208] and shows that the first-order terms uniquely determine a basic family.

THEOREM 5.7. *Let $s_1, s_2, \ldots,$ be any sequence in F with $s_1 \neq 0$. Then there exists a unique differential operator $G \in \text{Form}[L]$, such that for $Q_0, Q_1, \ldots$ (the basic family for G), $Q_i = s_i P_1 + \sum_{j=2}^{i} \ell_{ji} P_j$, where $\ell_{ji} \in F$.*

Proof. By Lemma 5.6, let $G \in \text{Form}[L]$ be s.t. $L = s_1 G + (s_2/2!) G^2 + (s_3/3!) G^3 + \cdots$. Observe that by Taylor's Theorem (Theorem 2.4(c)), $Q_i(x) = Q_i(0) + [(LQ_i)(0)/1!] P_1 + \cdots$. But $(LQ_i)(0) = \sum_{j=1}^{i} s_j [i^{(j)}/j!] Q_{i-j}(0) = s_i$, i.e., $Q_i = s_i P_1 + \cdots$. Uniqueness of G follows from Lemma 5.6 because of Theorem 3.1. Namely, if $Q_i = s_i P_1 + \cdots$, $(LQ_i)(0) = s_i$, then $L = \sum_{i=1}^{\infty} (s_i/i!) G^i$. ∎

REFERENCES

1. C. BERGE, "Principles of Combinatorics," Academic Press, New York, 1971.
2. J. P. FILLMORE AND S. G. WILLIAMSON, A linear algebra setting for the Rota–Mullin theory of polynomials of binomial type, *Lin. Multinear Alg.* 1 (1973), 67–80.
3. A. M. GARSIA, An expose of the Mullin–Rota theory of polynomials of binomial type, *Lin. Multilinear Alg.* 1 (1973), 47–65.
4. R. MULLIN AND G.-C. ROTA, On the foundations of combinatorial theory, III: Theory of binomial enumeration, *in* "Graph Theory and Its Applications" (B. Harris, Ed.), pp. 167–213, Academic Press, New York, 1970.
5. I. NIVEN, Formal power series, *Am. Math. Monthly* **76** (1969), 871–89.
6. G.-C. ROTA, D. KAHANER, AND A. ODLYZKO, On the foundations of combinatorial theory. VIII. Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973), 684–760.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.