THE NUMBER OF MAXIMAL SUBGROUPS OF THE SEMIGROUP OF BINARY RELATIONS II

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In this paper we extend the analysis of [4] to additional D-classes of the semigroup of binary relations. We will briefly discuss those results which we need from the preceeding paper. For more details, basic terminology and references see [4].

It is known (see [5]) that the maximal subgroups of a semigroup are the H-classes which contain an idempotent and that these H-classes lie in regular D-classes. Furthermore, it is known that there is a very direct relationship (see [4] for relevant terminology and references) between regular D-classes and distributive lattices. There is a natural correspondence between all distributive lattices of length k and all partially ordered sets of k elements (see [1]). Starting with a partially ordered set P of k elements one obtains a distributive lattice of length k by considering the set of all subsets of P which are "closed from below" ordered by inclusion. Furthermore, every distributive lattice can be obtained in this way.

Partially ordered sets can be associated with lower triangular Boolean relation matrices (see [3]). In [3], the first author studied certain classes of partially ordered sets, characterized by the number of interrelations which exist between the various elements of a given partially ordered set.

DEFINITION 1. Let n and k be natural numbers. By $P(n,k)$ we mean a set of representatives of isomorphism classes of partially ordered sets of n elements such that the partial ordering on the given set has $n+k$ elements. By $D(n,k)$ we mean the set of regular D-classes in B_n which correspond to the distributive lattices which are generated by the elements of $P(n,k)$. We use $P(n,k,i)$ and $D(n,k,i)$, $i=1, \cdots, |P(n,k)|$ to represent the individual elements of $P(n,k)$ and $D(n,k)$ respectively, with $D(n,k,i)$ corresponding to the lattices generated by $P(n,k,i)$, which we denote by $La\ (n,k,i)$. By $La\ (n,k)$ we denote the set of the $La\ (n,k,i)$. By $H(n,k,i)$ we mean the cardinality of an H-class (they are all the same) of $D(n,k,i)$.

REMARK. For $P(n,k)$, k may range anywhere between 0 and $n(n-1)/2$. For more details (see [3]).

The proof of the following theorem may be found in [3].

THEOREM 1. *Let $n \geqq 6$ be an integer and $p = n(n-1)/2$, then*

(*i*) $|P(n,0)|=1$, (*v*) $|P(n,p-2)|=(n-1)(n-2)/2$

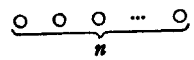
(*ii*) $|P(n,1)|=1$, (*vi*) $|P(n,p-1)|=n-1$

(*iii*) $|P(n,2)|=3$, (*vii*) $|P(n,p)|=1$.

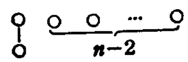
(*iv*) $|P(n,3)|=7$,

For reference and clarity we give the Hasse diagrams of the elements of $P(n, k)$, $k=0, 1, 2, 3, p-2, p-1, p$.

$P(n, 0, 1)$

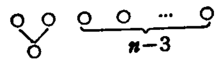


$P(n, 1, 1)$

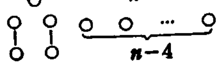


$P(n, 2, 1)$

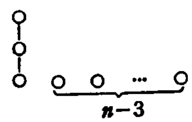
$P(n, 2, 2)$



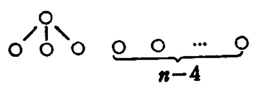
$P(n, 2, 3)$



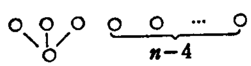
$P(n, 3, 1)$



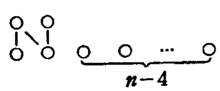
$P(n, 3, 2)$



$P(n, 3, 3)$

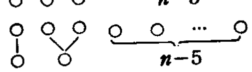


$P(n, 3, 4)$

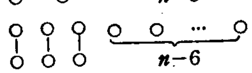


$P(n, 3, 5)$

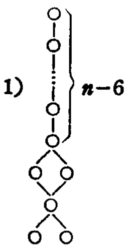
$P(n, 3, 6)$



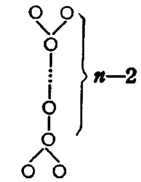
$P(n, 3, 7)$



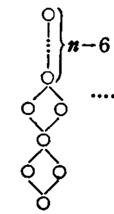
$P(n, p-2, 1)$



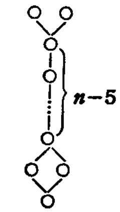
.... $P(n, p-2, n-4)$



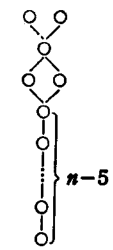
$P(n, p-2, n-3)$



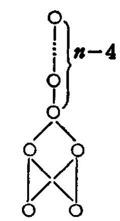
...... $P(n, p-2, 2n-9)$



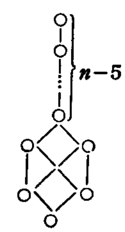
.... $P(n, p-2, (n-4)(n-3)/2)$



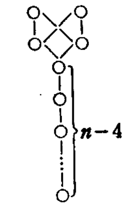
$P(n, p-2, (n-4)(n-3)/2+1)$



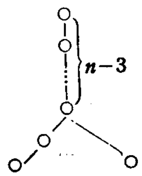
$P(n, p-2, (n-4)(n-3)/2+2)$



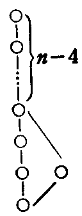
$P(n,\ p-2,\ (n-2)(n-3)/2)$



$P(n,\ p-2,\ (n-2)(n-3)/2+1)$

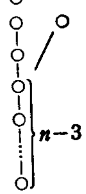


$P(n,\ p-2,\ (n-2)(n-3)/2+2)$

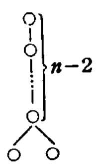


....

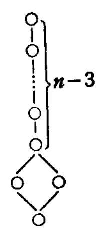
$P(n, p-2,\ (n-1)(n-2)/2)$



$P(n,\ p-1,\ 1)$

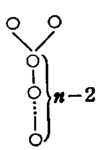


$P(n,\ p-1,\ 2)$

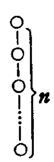


....

$P(n,\ p-1,\ n-1)$

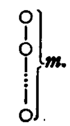


$P(n,\ p,\ 1)$



REMARK. In Theorem 1 above we insisted that $n \geqq 6$ so that all of the $D(n,k,i)$ exist for $k=0,\ 1,\ 2,\ 3,\ p-2,\ p-1,\ p$ $(i=1, \cdots, |P(n,k)|)$ and are distinct. For $n<6$ one need only make the proper corrections by omitting redundant or nonexistent partially ordered sets. Thus, suppose $n=3$, $P(n,2,3)$ cannot be realized with only 3 elements. If $n=3$, the only partially ordered sets which can be realized are versions of $P(3,0,1)$, $P(3,1,1)$, $P(3,2,1)$, $P(3,2,2)$, $P(3,3,1)$. In this paper we assume $n \geqq 6$, but the relevant modifications for $n<6$ can be made easily.

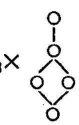
DEFINITION 2. Let $m \geqq 0$ be aninteger. By L_m we mean the lattice formed from all subsets of a set of m elements ordered by set inclusion. Thus $|L_m| = 2^m$. By C^m we mean the chain of m elements

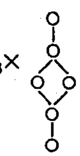


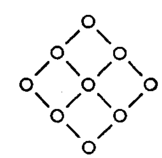
LEMMA 2. *For integer* $n \geqq 6$, *we have*

$La\ (n,0,1) \simeq L_n$

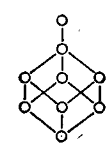
$La\ (n,1,1) \simeq L_{n-2} \times C_3$

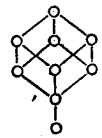
$La\ (n,2,1) \simeq L_{n-3} \times$ 

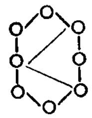
$La\ (n,2,2) \simeq L_{n-3} \times$ 

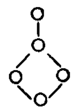
$La\ (n,2,3) \simeq L_{n-4} \quad \times$ 

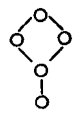
$La\ (n,3,1) \simeq L_{n-3} \times C_4$

$La\ (n,3,2) \simeq L_{n-4} \quad \times$ 

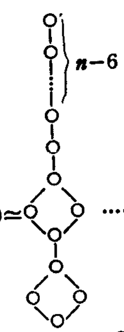
$La\ (n,3,3)\simeq L_{n-4}\times$ 

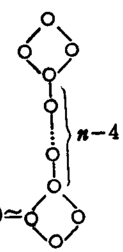
$La\ (n,3,4)\simeq L_{n-4}\times$ 

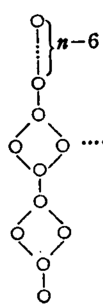
$La\ (n,3,5)\simeq L_{n-5}\times C_3\times$ 

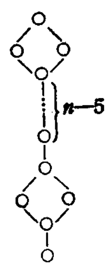
$La\ (n,3,6)\simeq L_{n-5}\times C_3\times$ 

$La\ (n,3,7)\simeq L_{n-6}\times C_3\times C_3\times C_3$

$La\ (n,p-2,\ 1)\simeq$ 

......$La\ (n,p-2,\ n-4)\simeq$ 

$La\ (n,\ p-2,\ n-3)\simeq$ 

......$La\ (n,\ p-2n-9)\simeq$ 

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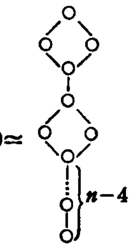
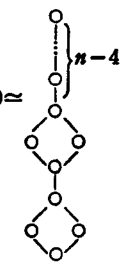
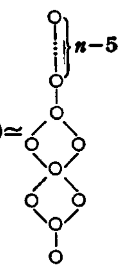
$La\ (n,\ p-2,\ (n-4)(n-3)/2)\simeq$ 

$La\ (n,p-2,(n-4)(n-3)/2+1)\simeq$ 

$La\ (n,p-2,(n-4)(n-3)/2+2)\simeq$ 

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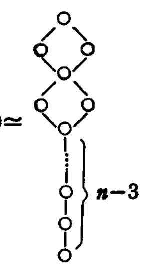
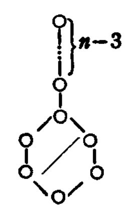
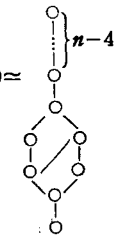
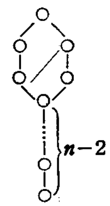
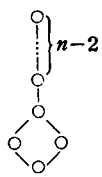
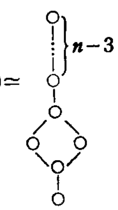
$La\ (n,\ p-2,\ (n-2)(n-3)/2)\simeq$ 

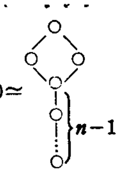
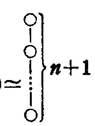
$La\ (n,\ p-2,\ (n-2)\ (n-3)/2+1)\ \simeq$ 

$La\,(n,\ p-2,\ (n-2)(n-3)/2+2) \simeq$  $\cdots\ \ \cdots$

$La\,(n,\ p-2,\ (n-1)(n-2)/2) \simeq$ 

$La\,(n,\ p-1, 1) \simeq$  $La\,(n,\ p-1, 2) \simeq$  $\cdots\cdot$

$\cdots\ \ \cdots$

$La\,(n,\ p-1,\ n-1) \simeq$  $La\,(n, p, 1) \simeq$ 

PROOF. The first 12 (up to and including $La\,(n, 3, 7)$) follow from the following observation. If one is trying to construct the distrbutive lattice of all the "closed from below" subsets of some given partially ordered set which is broken up into a number of mutually disjoint components, then by calculating the corresponding lattices for the various components and forming their lattice product, one gets a lattice isomorphic to the lattice generated by the given partially ordered set. It is not very difficult to prove this statement. It is easy to see how the first 12 calculations were made on the basis of this observation. The remaining lattices can be derived from the corresponding partially ordered sets by straightforward calculation, and we omit the details. We note that the cardinality of any element in $L(n, p-2)$ is $n+3$, while the cardinality of any element in $L(n, p-1)$ is $n+2$.

REMARK. For $n<6$ appropriate modification can be made in the listing of the lattices similar to the ones pointed out for the case of partially ordered sets in the remark preceding Definition 2.

We will now proceed to state several results which will be quite useful in attempting to calculate the number of maximal subgroups contained in certain regular D-classes.

THEOREM 3. *Let $A \in B_n$, then the Schutzenberger group associated with D_A is isomorphic to the group of lattice isomorphisms of $R(A)$.*

PROOF. See [2] for the proof.

The following is proved in [6].

THEOREM 4. *There is a natural isomorphism between the group of automorphisms of a finite distributive lattice L and the group of automorphisms of the partially ordered set formed by the join-irreducible elements of L.*

The two theorems above mean that the cardinality of any H-class in $D(n, k, i)$ is equal to the number of elements in the group of automorphisms of $P(n, k, i)$.

The following is proved and discussed in great detail in [6].

THEOREM 5. *Let $A \in B_n$ be regular, then the number of idempotents in D_A is equal to*

$$\Big(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (M(A)-i)^n\Big)/|H_A|$$

where k is the length of the lattice $R(A)$, and $M(A)$ is a number calculated as follows. one first identifies the meet-irreducible elements of $R(A)$. For each $w \in R(A)$ we let v_w be the join of all meet-irreducible elements which are not greater than or equal to w, and $q(w)$ be the number of elements in $R(A)$ which are greater than or equal to v_w. Then $M(A)=\sum_{w \in R(A)} q(w)$. In [6] it is shown that the dual definition also gives the same result for $M(A)$, i.e., let v_w be the meet of all join-irreducible elements which are not less than or equal to w, etc. We will use the dual definition in this paper.

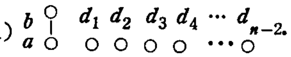
DEFINITION 3. Let $X \subset B_n$, then by $E(X)$ we mean the set of all *idempotents* in X.

Now we are ready to calculate the number of maximal subgroups in the D-classess corresponding to the lattices in Lemma 2. We know that this number is equal to the number of idempotents in the given D-class and so we get the following result.

THEOREM 6. *For $n \geqq 6$ we have*

k	i	$\lvert E(D(n,k,i))\rvert$	$H(n,k,i)$
0	1	$(\sum_{j=0}^{n}(-1)^j(2^{n+1}+n-1-j)^n/n!$	$n!$
1	1	$(\sum_{j=0}^{n}(-1)^j(3.2^{n-1}+n-j)^n)/(n-2)!$	$(n-2)!$
2	1	$(\sum_{j=0}^{n}(-1)^j(5\cdot2^{n-2}+n+2-j)^n)/2((n-3)!)$	$2((n-3)!)$
	2	$(\sum_{j=0}^{n}(-1)^j(5\cdot2^{n-2}+n+2-j)^n)/2((n-3)!)$	$2((n-3)!)$
	3	$(\sum_{j=0}^{n}(-1)^j(9\cdot2^{n-3}+n+1-j)^n)/2((n-4)!)$	$2((n-4)!)$
3	1	$(\sum_{j=0}^{n}(-1)^j(2^n+n+2-j)^n)/(n-3)!$	$(n-3)!$
	2	$(\sum_{j=0}^{n}(-1)^j(9\cdot2^{n-3}+n+6-j)^n)/6((n-4)!)$	$6((n-4)!)$
	3	$(\sum_{j=0}^{n}(-1)^j(9\cdot2^{n-3}+n+6-j)^n)/6((n-4)!)$	$6((n-4)!)$
	4	$(\sum_{j=0}^{n}(-1)^j(2^n+n+4-j)^n)/(n-4)!$	$(n-4)!$
	5	$(\sum_{j=0}^{n}(-1)^j(15\cdot2^{n-4}+n+2-j)^n)/2((n-5)!)$	$2((n-5)!)$
	6	$(\sum_{j=0}^{n}(-1)^j(15\cdot2^{n-4}+n+2-j)^n)/2((n-5)!)$	$2((n-5)!)$
	7	$(\sum_{j=0}^{n}(-1)^j(27\cdot2^{n-5}+n+2-j)^n)/6((n-6)!)$	$6((n-6)!)$
$P-2$	1 to $(n-2)\times(n-3)/2$	$(\sum_{j=0}^{n}(-1)^j((n^2+9n)/2-j)^n)/4$	4
	$(n-2)(n-3)2/+1$ to $(n-1)(n-2)/2$	$(\sum_{j=0}^{n}(-1)^j((n^2t+9n-6)/2-j)^n$	1
$p-1$	1 to $n-1$	$(\sum_{j=0}^{n}(-1)^j((n^2+7n)/2-j)^n)/2$	2
P	1	$\sum_{j=0}^{n}(-1)^j((n^2+5n+2)/2-j)^n$	1

PROOF. The proof is basically a matter of computation. Dual lattices and dual D-classes have the same number of idempotents corresponding to them, which cuts down the amount of work some. We will just give some examples of the calculation.

Consider $P(n,1,1)$ 

La $(n,1,1) \simeq L_{n-2} \times C_3$. We divide the "closed from below" subsets of $P(n,1,1,1)$ into three classes: (i) those which do not contain a: (ii) those which contains a but not b; (iii) those which contain b. The elements of each class form a sublattice of *La* $(n,1,1)$ isomorphic to L_{n-2}. Let w be any non-maximal element in class (i), then $q(w)=1$ since a and at least one of the d_i $(i=1, \cdots, n-2)$ are not less than or equal to w and hence $v_w=\phi$. Class (i) contains a single maximal element (the set of all the d_i's, $i=1, \cdots, n-2$) call it m. Clearly v_m is the meet of the set corresponding to b ({a, b}) and the set corresponding to a ($\{a\}$), namely $\{a\}$. Thus $q(m)=2$, since $\{a\}$ and ϕ are the only two elements of $La(n,1,1)$ less that or equal to $\{a\}$. Hence,

$$\sum_{w \in class\ (i)} q(w)=2^{n-2}+1.$$

In class (ii), $q(w)=1$ for all non-maximal w, but $q(m)=3$ for the unique maximal element m of class (ii). Hence,

$$\sum_{w \in class\ (ii)} q(w)=2^{n-2}+2.$$

Finally, in class (iii) $q(w)=1$ for all w which contain no more then $(n-4)$ of the d_i. For the $(n-2)$ w's which contain exactly $(n-3)$ of the d_i, we have that v_w is equal to $\{d_{i_w}\}$ where d_{i_w} is the only one of the d_i's not contained in w. Hence for these $(n-2)$ w's, $q(w)=2$. Finally, For the maximal element m of class (iii) which is also the maximal element of *La* $(n,1,1)$ we have that $v_m=m$ and hence $q(m)=\lvert La\ (n,1,1)\rvert=3\cdot2^{n-2}$. Thus

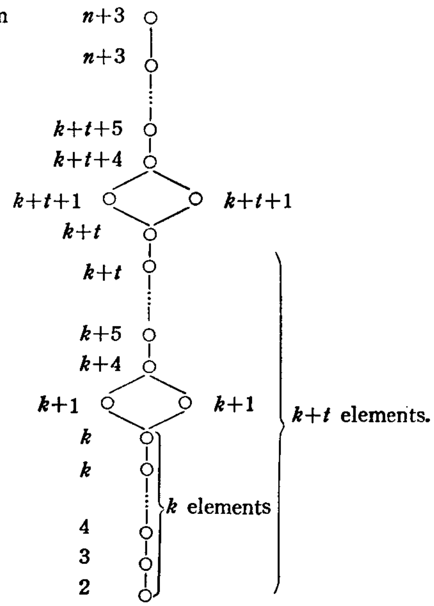
$$\sum_{w \in class\ (iii)} q(w)=(2^{n-2}-(n-1))+2(n-2)+3\cdot2^{n-2}$$
$$=4\cdot2^{n-2}+n-3.$$

Hence, for any $A \in D(n,1,1)$ we would have

$$M(A)=\sum_{w \in La\ (n,1,1)} q(w) =3\cdot2^{n-1}+n.$$

Clearly, $H(n,1,1)=(n-2)!$ by Theorems 3 and 4, and the rest of the result follows from Theorem 5.

To conclude, we give an idea of the calculations for $k=p-2$, $i=1, \cdots, (n-2)(n-3)/2$. For these values of k and i, it is easy to see that $H(n,k,i)=4$. Furthermore the lattices *La* (n,k,i) have the general form



The numbers next to the various points of the lattices are the value of q at that point. Adding up the values one gets $M(A)=(n^2+9n)/2$ for $A\in D(n,p-2,i)$ $(i=1, \cdots, (n-2)(n-3)/2)$. One can easily check that $(n^2+9)/2$ is the correct value for all cases, including limiting cases, such as the case where $k=1$.

The various other cases were all calculated in a similar manner, and we shall omit the details because the calculations are quite similar to the case we have discussed.

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