PROPAEDEUTIC TO CHAIN-COMPLETE POSETS WITH BASIS

George Markowsky

IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598

ABSTRACT

In this paper, several relations between members of a chain-complete poset (weaker than the relations relatively compact and IN introduced by Scott in his study of bases) are considered, with an eye toward examining whether all the properties of bases can be obtained from weaker axioms. It turns out that for all but one relation (relatively chain compact) one gets a weaker notion of bases than using relatively compact or IN. For relatively chain compact we conjecture that one gets a weaker notion of basis. However, we show that in the presence of bounded joins the existence of a basis derived from the relation of relatively chain compact implies the existence of a Scott basis if the cardinality of the chain-complete poset is less than the ω-th infinite cardinal. If one accepts the Continuum Hypothesis this means that the weaker relation is adequate in many posets likely to be of interest to computer science. Finally, even in the presence of the "weaker" type of basis in a complete lattice, the meet operation is continuous.

§ 0 Introduction

Following Dana Scott's lead [S1], a number of authors ([A],[EC],[M2],[MR],[R1],[R3],[SM],[V]) have examined the notion of basis for various classes of posets.

Definition 1 Let P be a poset and $x, y \in P$.

(i) x is said to be *relatively compact to y*, xRCy, if for every *nonempty* directed set $D \subseteq P$ with sup $D \geq y$, there exists $d \in D$ with $d \geq x$.

(ii) A *nonempty* directed set $D \subseteq P$ consisting of elements relatively compact to x is a *local basis* for x if sup $D = x$.

(iii) P has a *basis* if every element of P has a local basis. A subset B is a *basis of P* whenever for every $x \in P$ some subset of B is a local basis for x and if every element of B is in some local basis which we denote by B_x.

(iv) We use D_y to denote the set $\{x \in P \mid xRCy\}$.

(v) If xRCx, then we say x is *compact*. □

Definition 2 Let P be a poset and $x, y \in P$.

(i) x is said to be *relatively chain-compact (relatively chain-irreducible, relatively directed irreducible), xRCCy (xRCIy, if for every nonempty* chain (chain, directed set) in P with sup $D \geq y$ (sup $D = y$, sup $D = y$), there exists $d \in D$ with $d \geq x$.

(ii) A *nonempty* directed set $D \subseteq P$ consisting of elements relatively chain-compact (relatively chain-irreducible, relatively directed irreducible) to x is a *local CC-basis (local CI-basis, local DI-basis)* for x if sup $D = x$.

(iii) P is said to have a *CC-basis* (*CI-basis, DI-basis*) if every element of P has a local CC-basis (CI-basis, DI-basis). A subset B is a *CC-basis of P (CI-basis of P, DI-basis of P)* whenever for every $x \in P$ some subset of B is a local CC-(CI-,DI-) basis for x and if every element of B is in some local CC-basis (CI-basis, DI-basis) which we denote by B_x.

(iv) We use CCD_y (CID_y, DID_y) to denote $\{x \in P \mid xRCCy\}$ ($\{x \in P \mid xRCIy\}$, $\{x \in P \mid xRDIy\}$).

(v) If xRCCx (xRCIx,xRDIx) we say x is *chain-compact (chain-irreducible, directed-irreducible)*. □

In [MR; Lemma 2.6] it is shown that in a chain-complete poset xRCCx iff xRCx. In [M2] it is shown that x has a local basis iff D_x is a local basis. Similarly, x has a local DI-basis iff DID_x is a local DI-basis. This result is not true for local CC-bases and local CI-bases.

Theorem 3 Let P be a poset, $x, y \in P$, then: xRCy implies xRCCy, xRDIy and xRCIy; xRCCy implies RCIy; xRDIy implies xRCIy. Furthermore, if P has a basis, then it has a CC-basis,CI-basis and DI-basis. Similarly, if P has a CC-basis it has a CI-basis and a DI-basis is also a CI-basis. □

Example 4 shows that RC, RCC $\not\equiv$ RCI, RDI and Example 5 shows that RCI $\not\equiv$ RDI. Furthermore, none of these concepts are equivalent to the topological concept IN used by Scott ([S1],[S2]). However, it turns out ([M2], [S2]) that the notion of basis using the concept of IN is equivalent to the one using RC, i.e., the fact that each point y of a poset can be written as a sup of a directed set of elements "nicely" related to y strengthens the relationship. A similar situation occurred in [MR] where a basis defined using the weaker notion of chain-irreducible turned out to be equivalent to a basis defined using the notion of compactness. Thus it is of interest to determine the extent to which having a basis of a certain type actually gives one of a stronger type.

It will be shown that the set {basis, CC-basis, DI-basis, CI-basis} includes at least three distinct concepts. The only ambiguity is over the equivalence of basis and CC-basis. We conjecture that basis $\not\equiv$ CC-basis, but do not have a counterexample in hand. A counterexample might be fairly difficult, since Theorem 12 shows that basis $\equiv$ CC-basis if P has bounded joins and $|P| < \omega_{\omega}$ (where ω_i denotes the i-th infinite cardinal number). Furthermore, given any nonempty directed set D and a point y and $y \leq$ sup D the set of elements of CCD_y lying below some element of D is extensive enough to have y as its sup (Theorem 11). This coupled with the fact (Theorem 8) that meet is continuous in a complete lattice with a CC-basis might make a CC-basis enough of an assumption even if basis $\not\equiv$ CC-basis.

§ 1 DI-basis $\not\equiv$ CC-basis, basis $\not\equiv$ CI-basis

Example 4 Let N^∞ be the natural numbers together with ∞ ordered in the obvious manner. Let $P=\{0,1\}\times N^\infty$ be ordered by

$(a,m)\leq(b,m')$ iff (i) $a=b$ and $m\leq m'$ in N^∞

or

(ii) $(a,m)=(0,0)$

or

(iii) $(b,m')=(1,\infty)$

Thus P looks like

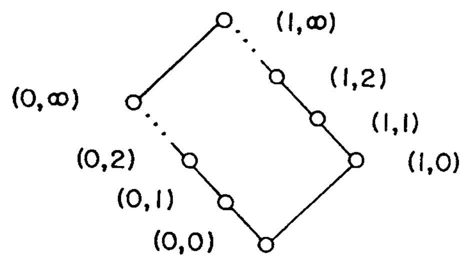

Figure 1

Note that P is chain-complete, but lacks a CC-basis because $yRCC(0,\infty)$ implies that $y=(0,0)$. To see this, note that $C^*=\{(1,m) \mid m\in N^\infty-\{\infty\}\}$ is a chain with sup $C^*=(1,\infty)>(0,\infty)$, but the only element less than $(0,\infty)$ and less than some element of C^* is $(0,0)$. Since sup $\{(0,0)\}=(0,0)\neq(0,\infty)$, P lacks a CC-basis.

That P has a DI-basis follows from the following observations.

(1) Every element of $B=P-\{(0,\infty),(1,\infty)\}$ is directed-irreducible.

(2) In B, every directed set is a chain.

(3) For $y\in B$, $DID_y=\{x\leq y\}$, $DID_{(0,\infty)}=\{(0,m) \mid m\in N\}$ and $DID_{(1,\infty)}=\{(1,m)\mid m\in N\}\cup\{(0,0)\}$. □

Note that P is a lattice so that even for complete lattices, DI-basis $\not\equiv$ CC-basis. From Theorem 3, it follows that basis, CC-basis $\not\equiv$ DI-basis, CI-basis.

§ 2 CI-basis $\not\equiv$ DI-basis

Example 5 Let ω be the first infinite ordinal and ω_1 the first uncountable ordinal. As usual, we consider an ordinal to be the set of all its predecessors. Thus $\omega+1=\omega\cup\{\omega\}$ and $\omega_1+1=\omega_1\cup\{\omega_1\}$.

Let $P=((\omega_1+1)\times(\omega+1)\times\{0,1\})-\{(\omega_1,\omega,0)\}$, be ordered as follows:

$(a,b,m)\leq(a',b',m')$ iff (i) $m=m'$, $a\leq a'$ and $b\leq b'$

or

(ii) $m=0$, $m'=1$, $a'=\omega_1$ and $b\leq b'$

or

(iii) $m=0$, $m'=1$, $a\leq a'$ and $b'=\omega$

or

(iv) $(a,b,m)=(0,0,0)$, where the orderings on the components are the ordinal ordering.

One can verify directly that P is a complete lattice. Let $D=\omega_1\times\omega\times\{0\}$ and $D'=\omega_1\times\omega\times\{1\}$. Both D and D_1 are directed subsets of P and have $(\omega_1,\omega,1)$ as their sup. Thus if $yRDI(\omega_1,\omega,1)$, then $y\leq x,x'$ for some $x\in D$, $x'\in D$. Hence $y=(0,0,0)$. It follows that $DID_{(\omega_1,\omega,1)}=\{(0,0,0)\}$ and that P lacks a DI-basis.

The fact that P has a CI-basis follows from the following observations.

(1) Any chain $C\subseteq P$ with sup $C=(\omega_1,\omega,1)$ must contain $(\omega_1,\omega,1)$ or a cofinal subset of one of the following four chains: $\{\omega_1\}\times\omega\times\{0\}$; $\{\omega_1\}\times\omega\times\{1\}$; $\omega_1\times\{\omega\}\times\{0\}$; $\omega_1\times\{\omega\}\times\{1\}$. To see this note that any chain in P with sup $=(\omega_1,\omega,1)$, not containing $(\omega_1,\omega,1)$, and not cofinal in any of the four chains mentioned above is contained entirely in D or D'. There are two cases to consider. If the chain is countable then the first components form a countable subsequences of ω_1 and cannot possibly have ω_1 as a limit. If the chain is uncountable, there exist $n\in\omega$ such that an uncountable number of elements have their second component $=n$. But this implies that some element of the chain has its first component $=\omega_1$, contradicting the assumption that the chain was entirely contained in D or D'.

(2) $CID_{(\omega_1,\omega,1)}=D$.

(3) For $y=(a,b,0)$, $CID_y=\{(c,d,0)\mid c\leq^*a, d\leq^*b\}$ where for two ordinals α,β $\alpha\leq^*\beta$ means either that β is a limit ordinal and $\alpha<\beta$ or that β is not a limit ordinal and $\alpha\leq\beta$.

(4) For $y=(a,b,1)$ with $(a,b)\neq(\omega_1,\omega)$, $CID_y=\{(c,d,1)\mid c\leq^*a, d\leq^*b\}\cup\{(0,0,0)\}$, since no chain in D can have y a a sup and thus no element of D' needs to be $\leq$ any element of D. On the other hand, there is always a chain in D' having y as a sup which shows that *nonzero* elements from D cannot be RCIy. □

Thus even for complete lattices CI-basis $\not\equiv$ DI-basis.

§ 3 Is Basis $\not\equiv$ CC-basis?

In section 4 of [S1], Scott notes that the existence of a basis in a complete lattice implies that the meet operation is continuous, i.e., it distributes over taking sups of directed sets. In this section, we shall show that the meet is continuous whenever the lattice has a CC-basis. Furthermore, the existence of CC-basis is equivalent to the meet being continuous and the existence of a CI-basis. Similarly, the existence of a basis is equivalent to the continuity of the meet and the existence of DI-basis. The final results in this section illustrate some of the problems in constructing a counterexample to basis $\not\equiv$ CC-basis. For instance, if P is a poset with a CC-basis, B, $y \in P$ and $D^* \subseteq P$ is a nonempty directed set with $y \leq \sup D^*$ then there are always enough elements RCCy such that their sup is y.

The proofs of the following two lemmas can be found in [M1: Theorem 1 and Corollary 3].

Lemma 6 (Sharpened Iwamura's Lemma) If D is an infinite directed set, then there exists a transfinite sequence D_α, $\alpha < |D|$, of directed subsets of D having the following properties:

(1) for each α, if α is finite, so is D_α, while if α is infinite $|D_\alpha| = |\alpha|$ (thus for all α, $|D_\alpha| < |D|$);

(2) if $\alpha < \beta < |D|$, $D_\alpha \subseteq D_\beta$;

(3) $D = \cup_\alpha D_\alpha$. □

Lemma 7 If P is a chain-complete poset and f: P→P preserves sup's of nonempty chains, it preserves sup's of nonempty directed sets. □

Theorem 8 Let P be a complete lattice with a CC-basis, then the meet operation is continuous.

Proof: By Lemma 7 it is enough to show that for any $x \in P$ and nonempty chain $\{y_\alpha\} \subseteq P$ with $y = \sup\{y_\alpha\}$, $x \wedge y = \sup\{x \wedge y_\alpha\}$. Clearly, $x \wedge y \geq \sup\{x \wedge y_\alpha\}$. Let $z \in CCD_{x \wedge y}$, then for some y_α, $z \leq y_\alpha$, whence $z \leq x \wedge y_\alpha$. Since $x \wedge y = \sup CCD_{x \wedge y} \leq \sup \{x \wedge y_\alpha\}$, we are done. □

Corollary 9 Let P be a complete lattice. Then the following equivalent.

(1) P has a CC-basis.

(2) P has a CI-basis and the meet operation is continuous.

Proof: Theorem 8 shows that (1) implies (2). We claim that (2) implies that RCC=RCI, whence (1) follows. Let C be a nonempty chain in P and $x, y \in P$ such that xRCIy and $y \leq \sup C$. By continuity of $\wedge$, $x = \sup\{y \wedge t \mid t \in C\}$ whence for some $t \in C$, $t \geq y \wedge t \geq x$, i.e., xRCCy. □

The proof of the following corollary is easier than that of Corollary 9, since we do not need to use Lemma 7.

Corollary 10 Let P be a complete lattice. Then the following are equivalent.

(1) P has a basis.

(2) P has a DI-basis and meet is continuous. □

The above results show that as far as having lattice operations be continuous, a CC-basis is as good as a basis. It remains to investigate the relationship between the concepts of CC-basis and basis.

Theorem 11 Let P be a chain-complete poset with a CC-basis P.

(a) If $y \in P$ and $D \subseteq P$ is a nonempty directed set with $y \leq \sup D$, then if $CCD_{y,D} = \{xRCCy \mid x \leq d$ for some $d \in D\}$, $CCD_{y,D} \neq \emptyset$ and $y = \sup CCD_{y,D}$.

(b) If P has bounded joins, then CCD_y is a local CC-basis and $CCD_{y,D}$ is a directed set.

Proof: a) The proof is by induction on $|D|$. Assume D is countable, then it contains a cofinal chain and $CCD_{y,D} = CCD_y$. Assume the theorem holds for all nonempty directed sets D with $|D| < \lambda$. Let D' be a directed set with $|D'| = \lambda$. By Lemma 6 there exists a transfinite nested sequence of directed sets $\{D_\alpha\}_{\alpha<\lambda}$ with $|D_\alpha| < \lambda$ and $D' = \cup D_\alpha$.

Let $z_\alpha = \sup D_\alpha$ for $\alpha < \lambda$. $\{z_\alpha\}_{\alpha<\lambda}$ is a chain with $\sup\{z_\alpha\}_{\alpha<\lambda} = \sup D \geq y$. For each $x \in CCD_y$, there exists α_x with $x \leq z_{\alpha_x}$. By induction, $CCD_{x,D_{\alpha_x}} \neq \emptyset$ and has x as its sup. However, $CCD_{y,D} \supseteq \bigcup_{xRCCy} CCD_{x,D_{\alpha_x}}$.

(b) If x_1RCCy and x_2RCCy, $x_1 \vee x_2$ exists and is RCCy. Thus CCD_y is a directed set. Similarly, if $x_1, x_2 \in CCD_{y,D}$, $x_1 \vee x_2 \in CCD_{y,D}$ since D is directed. □

Our final result shows that for a poset with a CC-basis and a "small enough" cardinality, i.e., $<\omega_\omega$, there are enough relatively compact elements to enable one to reconstruct each element of the poset by taking sups. If P has bounded joins, CC-basis is ≡ basis. This helps to indicate some of the complexity necessary to take into account in trying to construct a counterexample.

Theorem 12 Let P be a chain-complete poset with a CC-basis, B, and $|P| < \omega_\omega$, then for all $y \in P$ $y = \sup D_y$. If P has bounded joins, D_y is directed and P has a basis.

Proof: For $y \in P$, define $B_{y,0} = B_y$ and $B_{y,i+1} = \{z \in B_{x,i} \mid x \in B_y\}$ for $i = 1, 2, \ldots$. It is easy to see that $\sup B_{y,i} = y$ for all $i \in \mathbf{N}$.

We claim that $|P| = \omega_i$ implies $B_{y,i} \subseteq D_y$. Actually, a somewhat stronger result holds, namely, if $|D| = \omega_i$ is a directed subset of P with $\sup D \geq y$, then for all $x \in B_{y,j}$, there exists $d \in D$ with $x \leq d$. For countable D the result is obvious since D has a cofinal chain. For larger D, the result follows from Lemma 6 in the usual fashion. □

References

[A] Astesiano, E., "Bases for Scott-continuous Posets," Universita di Genova, Pubblicazioni dell' Instituto di Matematica No. (181) 1976.

[EC] Egli, H. and Constable, R. L., "Computability Concepts for Programming Language Semantics," *Theoretical Comp. Science*, 2 (1976) 133-145.

[M1] Markowsky, G., "Chain-complete Posets and Directed Sets with Applications," *Algebra Universalis* 6 (1976) 53-68.

[M2] Markowsky, G., "A Motivation and Generalization of Scott's Notion of a Continuous Lattice," These Proceedings.

[MR] Markowsky, G. and Rosen, B. K., "Bases for Chain-complete Posets," *IBM J. of Res. and Dev.*, 20 (March, 1976) 138-147.

[R1] Reynolds, J. "Notes on a Lattice-theoretic Approach to the Theory of Computation," Lecture Notes, Syracuse U., October, 1972.

[R2] Reynolds J., "On the Interpretation of Scott Domains" *Symp. Math.* 15 (1975) 123-135.

[S1] Scott, D., "Outline of a Mathematical Theory of Computation," *Proc. 4th Ann. Princeton Conf. on Inform. Sci. and Systems*, 1970, 169-176.

[S2] Scott, D., "Continuous Lattices," *Proc. Dalhousie Conf. on Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972, 97-136.

[S3] Scott, D., "Data Types as Lattices," *SIAM J. Comput.* 5 (1976) 522-587.

[SM] Smyth, M., "Effectively Given Domains," *Theoretical Comp. Science* 5 (1977) 257-274.

[V] Vuillemin, J., *Syntaxe, Semantique, et Axiomatique d'un Language de Programmation Simple*, These d'etat, University of Paris, Sept. 1974.