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Extending semilattices is hard

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Abstract. We show that the semilattice extension problem raised by Arbib and Manes in [2] is *NP*-hard, implying that no simple solution is likely to be forthcoming.

In [2], Arbib and Manes define a *composite algebra* to be a complete lattice together with a finite number of sup-preserving endomorphisms. They then specialize the problem as follows: let X^* be the free monoid on the finite set X, Y a lattice and Y^{X^*} be the composite algebra with pointwise sups and a family of sup-preserving endomorphisms given by $\varphi_x(f) = fL_x$ for each $x \in X$, where for each word $w \in X^*$, $L_x w = xw$.

They then pose the following problem. Let $(A, \{\varphi_x \mid x \in X\})$ be the join-closure of the X-closure of a single element f of Y^{X^*}. Assuming that A is finite, find a composite subalgebra $(A^*, \{\varphi_x \mid x \in X\})$ containing A and having the smallest number of join-irreducibles (note A is a join-sublattice of A^* which is a join-sublattice of Y^{X^*}). We will show that this problem is at least as hard as the set basis problem which Larry Stockmeyer [3] has shown to be *NP*-complete. Thus we are not likely to find a simple solution to the Arbib-Manes problem (see the discussion in Chapter 10 of [1]), since *NP*-complete problems (e.g. finding Hamiltonian Paths) have been around for a long time, without anyone coming up with any "simple" solutions. Most people working in the field of algorithms take this as evidence for the thesis that these problems are "intrinsically" hard.

The set basis problem is the following. Given a set U of cardinality n and nonempty subsets $S_1, \ldots, S_m$ of U, decide whether for a given integer k we can find c nonempty subsets $T_1, \ldots, T_c$ of U with $c \le k$, such that each S_i is a union of some of the T_j's. We shall now show that a general method for solving the Arbib–Manes problem would yield a technique for finding a set basis of smallest cardinality for an arbitrary family of sets $\{S_i\}_{i=1}$.

Given U and the nonempty subsets $S_1, \ldots, S_m$ let $X^* = \{x_1, \ldots, x_{m-1}\}$ be an $m-1$ element set, $Y = 2^U$ and $f: X^* \to Y$ be given by $f(\lambda) = S_m$ (where λ is the empty word) and $f(wx_i) = S_i$ for all i and $w \in X^*$. Note that $\varphi_{x_i} f(w) = f(w)$ for all i

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and nonempty words and that $\varphi_{x_i}\varphi_{x_j}(f)=\varphi_{x_i}(f)$ for all i, j. Thus the X-closure of $\{f\}$, A', is finite and so is, A, the join-closure of A'.

Let A^*, $\{\varphi_x \mid x\in X\}$ be any composite subalgebra of Y^{x^*} containing A and let $g_1,\ldots,g_r$ be the join irreducibles of A^*. Since each of $f, \varphi_{x_1}(f),\ldots,\varphi_{x_{m-1}}(f)$ is a union of the g_j's, each $S_i=\varphi_{x_i}(f)\lambda$ $(i=1,\ldots,m-1)$ and $S_m=f(\lambda)$ must be the union of the $g_j(\lambda)$'s. Thus $r\geq k$ where k is the smallest set-basis for the S_i.

Conversely, suppose $T_1,\ldots,T_k$ is the smallest set basis for the S_i. Let $g_j: X^*\to Y$ be given by $g_j(\lambda)=T_j$, $g_j(w)=f(w)$ for all $j=1,\ldots,k$ and nonempty w. Let $B=\{g_j\}_{j=1,\ldots,k}$. Note that $\bar{B}_{\text{def}}=B\cup(\bigcup_{i=1}^{m}\varphi_{x_i}(B))=B\cup\{\varphi_{x_i}(f)\mid i=1,\ldots,m\}$. Thus, $\varphi_{x_i}(\bar{B})\subseteq\bar{B}$ for all i. Since f and each $\varphi_{x_i}(f)$ is a union of the g_j's, the join-closure of $\bar{B}$ is a composite subalgebra of Y^{X^*} which contains A and has no more than k join-irreducibles.

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