Order 9: 265–290, 1992.

Primes, Irreducibles and Extremal Lattices

GEORGE MARKOWSKY
Computer Science Department, University of Maine, Orono, ME 04469, U.S.A.

Communicated by I. Rival

(Received: 16 October 1991; accepted: 20 October 1992)

Abstract. This paper studies certain types of join and meet-irreducibles called coprimes and primes. These elements can be used to characterize certain types of lattices. For example, a lattice is distributive if and only if every join-irreducible is coprime. Similarly, a lattice is meet-pseudocomplemented if and only if each atom is coprime. Furthermore, these elements naturally decompose lattices into sublattices so that often properties of the original lattice can be deduced from properties of the sublattice. Not every lattice has primes and coprimes. This paper shows that lattices which are "long enough" must have primes and coprimes and that these elements and the resulting decompositions can be used to study such lattices.

The length of every finite lattice is bounded above by the minimum of the number of meet-irreducibles (meet-rank) and the number of join-irreducibles (join-rank) that it has. This paper studies lattices for which length = join-rank or length = meet-rank. These are called p-extremal lattices and they have interesting decompositions and properties. For example, ranked, p-extremal lattices are either lower locally distributive (join-rank = length), upper locally distributive (meet-rank = length) or distributive (join-rank = meet-rank = length). In the absence of the Jordan–Dedekind chain condition, p-extremal lattices still have many interesting properties. Of special interest are the lattices that satisfy both equalities. Such lattices are called extremal; this class includes distributive lattices and the associativity lattices of Tamari. Even though they have interesting decompositions, extremal lattices cannot be characterized algebraically since any finite lattice can be embedded as a subinterval into an extremal lattice. This paper shows how prime and coprime elements, and the poset of irreducibles can be used to analyze p-extremal and other types of lattices.

The results presented in this paper are used to deduce many key properties of the Tamari lattices. These lattices behave much like distributive lattices even though they violate the Jordan–Dedekind chain condition very strongly having maximal chains that vary in length from $N - 1$ to $N(N - 1)/2$ where N is a parameter used in the construction of these lattices.

Mathematics Subject Classification (1991). 06B05.

Key words. Join-irreducible, meet-irreducible, prime, coprime, extremal, distributive, locally distributive, Jordan–Dedekind chain condition, retract, poset of irreducibles, context, Tamari lattices.

1. Introduction

This paper is motivated by the combinatorial characterization of distributive lattices found in ([1] Theorem 4.6) and [17]: a finite lattice is distributive if and only if it satisfies the Jordan–Dedekind chain condition and the length of any maximal chain is equal to the number of join-irreducibles and equal to the number of meet-irreducibles. This result is generalized in ([1] Theorem 5.5) and [14] ([20]) to characterize locally distributive lattices. The above results make no statements about lattices that do not satisfy the Jordan–Dedekind chain condition.

The concept of a prime ideal in a lattice is widely known ([5, 6]). Less well known, but closely related are the concepts of *coprime* and *prime* elements in a lattice. These are special types of join-irreducibles and meet-irreducibles that have been used by a variety of authors ([3, 10, 11] [25], p. 51, [26]). Finite distributive lattices are characterized by the property that every join-irreducible is coprime, or dually that every meet-irreducible is prime. Finite meet-pseudocomplemented (join-pseudocomplemented) lattices are characterized by the property that every atom (coatom) is coprime (prime). Coprime/prime pairs permit the decomposition of a lattice into a disjoint union of two lattices related by a join-homomorphism/meet-homomorphism pair. The poset of completely prime elements in a complete lattice is isomorphic to the poset of completely coprime elements. This result generalizes the result that in a distributive lattice the poset of join-irreducibles is isomorphic to the poset of meet-irreducibles.

This paper introduces three classes of finite lattices that have primes and coprimes: *join-extremal lattices*, *meet-extremal lattices* and *extremal lattices*. Join-extremal lattices are lattices whose length is equal to the number of join-irreducibles in the lattice. Meet-extremal lattices are defined dually. Finally, extremal lattices are those lattices that are both join-extremal and meet-extremal. These three classes of lattices have coprime/prime decompositions where one of the factors is of the same type as the original lattice.

These three classes of lattices will be collectively called *p-extremal lattices*. *P*-extremal lattices include distributive lattices, locally distribututive lattices, and Tamari Associativity lattices (see Section 7) as special cases.

P-extremal lattices can be characterized by their *posets* of *irreducibles*. This characterization shows that arbitrary finite lattices can be embedded into finite extremal lattices, so there are no algebraic characterizations of *p*-extremal lattices. Nevertheless, *p*-extremal lattices have many interesting properties. These ideas also lead to interesting decompositions of distributive and locally distributive lattices into two sublattices.

The above results provide insight into the structure of the Tamari Associativity lattices, T_n, which consist of all possible parenthesizations of $(n+1)$ factors. T_n is a complemented, pseudocomplemented, semidistributive, extremal lattice having length $n(n-1)/2$, but a shortest maximal chain of length $n-1$. Furthermore, T_{n-1} is a strong retract of T_n and a sublattice of its complement in T_n.

2. Preliminaries

Some of the results in this paper can be generalized to infinite lattices such as complete lattices or lattices with different types of chain conditions. This paper discusses some of these generalizations, but the primary focus is on finite lattices. To avoid trivialities, we assume that unless otherwise noted all lattices have at least two elements. All terms undefined in this paper can be found in [6].

DEFINITION 1. A *completely join-irreducible* element x of a lattice L is an element such that $x = \sup S$ implies that $x \in S$. A *completely meet-irreducible* element x of a lattice L is an element such that $x = \inf S$ implies that $x \in S$.

The *join-rank* of a lattice L, denoted by $jr(L)$ is the number of join-irreducible elements in L. Similarly, the *meet-rank* of a lattice L, denoted by $mr(L)$ is the number of meet-irreducible elements in L. ■

NOTATION 1. Let L be a lattice and $x \in L$. $J(L)$ ($M(L)$) denotes the set of all join-irreducible (meet-irreducible) elements of L, and $J(x)$ ($M(x)$) denotes $\{j \in J(L) \mid j \leqslant x\}$ ($\{m \in L \ m \geqslant x\}$). ■

The proof of the following simple lemma is left to the reader.

LEMMA 1. *Let L be a lattice of finite length. For each $x \in L$, $x = \sup J(x)$ and $x = \inf M(x)$. Also, length $(L) \leqslant |J(L)|, |M(L)|$.* ■

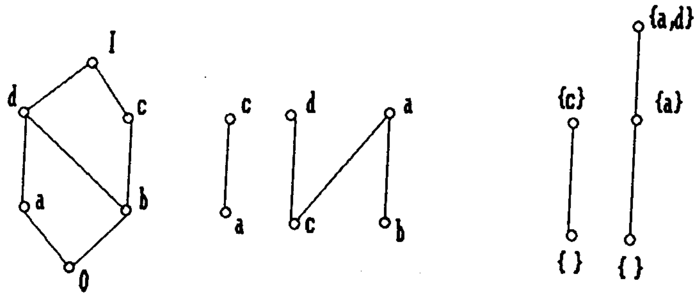
One goal of this paper is to characterize the cases where length$(L) = |J|$ or $|M|$. Since lattices satisfying one or the other of these equalities are as long as possible for the given join-rank or meet-rank, the names join-extremal and meet-extremal are appropriate. As noted in the introduction, *p*-extremal lattices have been completely characterized in the presence of the Jordan–Dedekind chain condition.

A key tool for obtaining the results in this paper is the poset of irreducibles introduced in [17] (see [19] and [21]). For finite lattices, some key facts are summarized below. The dual of the poset of irreducibles is used by the Darmstadt school and called a context (see [27]).

DEFINITION 2. (a) By a *bipartite directed graph* (*bidigraph*), D, we mean a triple (X, Y, Arcs) where X and Y are sets and $\text{Arcs} \subseteq X \times Y$. If $S \subseteq X$, $Ou(S) = \{y \in Y \mid \exists x \in S$ such that $(x, y) \in \text{Arcs}\}$. If $T \subseteq Y$, the $\text{In}(T) = \{x \in X \mid \exists y \in T$ such that $(x, y) \in \text{Arcs}\}$. For singleton sets we write $Ou(x)$ rather than $Ou(\{x\})$ and $\text{In}(y)$ rather than $\text{In}(\{y\})$. We will write $\text{Arcs}(D)$, $X(D)$, $Y(D)$ if necessary to reduce ambiguity.

(b) Let L be a finite lattice. The *poset of irreducibles* of L, $P(L)$ is the bidigraph $(J(L), M(L), \text{Arcs})$ where $(j, m) \in \text{Arcs}$ if and only if $j \nleqslant m$ in L. ■

THEOREM 1. ([19] Theorems 6, 15]. *Let L be a finite lattice and $P(L)$ its poset of irreducibles. Let $\Gamma(L) = \{Ou(S) \mid S \subseteq J(L)\}$. $\Gamma(L)$ is a lattice when ordered by set inclusion, and in $\Gamma(L)$ join corresponds to union. The map $f: L \to \Gamma(L)$ given by $f(x) = Ou(J(x))$ is a lattice isomorphism. Also, the group of the bidigraph automorphisms of $P(L)$ is isomorphic to the group of lattice automorphisms of $\Gamma(L)$. Finally, the disjoint components of $P(L)$ are isomorphic to the posets of irreducibles of the Cartesian factors of L.* ■



(i) L (ii) P(L) (iii) Factors of L

Fig. 1. An example of the poset of irreducibles.

REMARK 1. It is easy to verify that if $L^\wedge$ is the dual lattice of L, $P(L^\wedge)$ is just $P(L)$ turned upside down. The following theorem shows that every bidigraph generates a lattice and characterizes those bidigraphs that are exactly $P(L)$ for some lattice L. We are only interested in the finite case here. For infinite lattices see [17] and [19]. ■

THEOREM 2. ([19] Theorem 9). *Let $D = (X, Y, Arcs)$ be a finite bidigraph. D is isomorphic to $P(L)$ for some L if and only if the following holds:*

(i) *For all $x \in X$, if $W \subseteq X$ is such that $Ou(x) = Ou(W)$, then $x \in W$.*

(ii) *For all $y \in Y$, if $V \subseteq Y$ is such that $In(y) = In(V)$, then $y \in V$.* ■

EXAMPLE 1. Figure 1 shows a small lattice L and its associated $P(L)$. Note that $P(L)$ has two components corresponding to the posets of irreducibles of the two Cartesian factors of L. The reader can easily check that L is the Cartesian product of a 2 element chain and a 3 element chain.

The lattice L in Figure 1 happens to be a distributive lattice. An element, such as a or c in Figure 1, can be both join-irreducible and meet-irreducible. In such cases we can draw it twice, once in the bottom row of join-irreducibles and once in the top row of meet-irreducibles. For additional examples of $P(L)$ see [17, 18] or [19].

THEOREM 3. *A finite Cartesian of finite lattices is (join-, meet-) extremal iff every factor is (join-, meet-) extremal.*

Proof. We will give the proof only for join-extremal lattices. Theorem 1 implies that the number of join-irreducibles in a Cartesian product of lattices is just the sum of the number of join-irreducibles in each factor. Also, the length of a product is just the sum of the lengths of the factors. The results now follow from Lemma 1. ■

3. Coprime/Prime Decompositions of Lattices

DEFINITION 3. Let L be a lattice and a be in L. $a \neq I$ is called *prime* if for all x, y in L, $x \wedge y \leqslant a$ implies that $x \leqslant a$ or $y \leqslant a$. An element a is called *completely prime* if whenever $\inf S \leqslant a$, there exists $x \in S$ such that $x \leqslant a$. *Coprime* and *completely coprime* are defined dually.

Let L be a lattice with bounds O and I. An element $p \in L$ is said to have a *meet-pseudocomplement* (*join-pseudocomplement*) iff there exists an element p_* (p^*) such that $\forall q \in L$, $q \wedge p = O$ ($q \vee p = I$) iff $q \leqslant p_*$ ($q \geqslant p^*$). A lattice is said to be *meet-pseudocomplemented* (*join-pseudocomplemented*) iff every element has a meet-pseudocomplement (join-psuedocomplement). Following [8], we will call a lattice *pseudocomplemented* if it is both meet-pseudocomplemented and join-pseudocomplemented.

A lattice, L is said to be *meet-semidistributive* (*join-semidistributive*) iff for all $a, b, c \in L$, $a \wedge b = a \wedge c$ ($a \vee b = a \vee c$) implies $a \wedge (b \vee c) = a \wedge b$ ($a \vee (b \wedge c) = a \vee b$). A lattice is *semidistributive* iff it is both meet-semidistributive and join-semidistributive. ■

For lattices of finite length there is no difference between prime and completely prime elements, or between coprime and completely coprime elements. The proof of the following simple result is left to the reader.

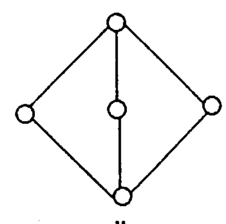
THEOREM 4. *If a is (completely) prime in L then a is (completely) meet-irreducible. If a is (completely coprime in L then a is (completely) join-irreducible. An element a of L is prime iff (a) is a prime ideal.* ■

REMARK 2. For finite lattices meet-semidistributivity implies meet-pseudocomplemention and dually. Also, Definition 3 differs from the definition in ([25] p. 51) in that the terms prime and coprime are used instead of meet-prime and join-prime, and that primes are $\neq I$ and coprimes are $\neq O$.

In a Cartesian product of complete lattices, the completely coprime (prime) elements are those having one component completely coprime (prime) while the other components are O (I).

One can characterize the coprime elements (and dually the prime elements) in terms of the poset of irreducibles as follows. $j \in J(L)$ is coprime iff $\exists m \in M(L)$ such that $m \in Ou(j)$ and $\forall j' \in J(L)$, $m \in Ou(j')$ implies that $Ou(j) \subseteq Ou(j')$. This proof is left to the reader.

The modular lattice M_5



M_5

shows that it is possible to have join-irreducibles that are not coprime and meet-irreducibles that are not prime. ■

The following result shows that primes and coprimes can be used to characterize finite distributive lattices and pseudocomplemented lattices. Theorem 5 together with Remark 2 provide a simple test that can determine whether a given finite lattice is pseudocomplemented.

THEOREM 5. (a) *Let L be a lattice such that every element is the join of join-irreducibles and the meet of meet-irreducibles, then the following are equivalent.*

(i) *Every join-irreducible is coprime.*

(ii) *Every meet-irreducible is prime.*

(iii) *L is distributive.*

(b) *A finite lattice is meet-pseudocomplemented iff each atom is coprime.*

(c) *A finite lattice is join-pseudocomplemented iff each coatom is prime.*

Proof. (a) This can be proved using the ideas in ([6] p. 59; Theorem 3). Variations of this theorem have been proved by ([2] Theorem 2) and ([26] Theorem 5).

(b) & (c) ([8] Theorem 3.3) shows that a lattice is meet-pseudocomplemented iff each atom, a, has a meet-pseudocomplement, a_*. The result follows if we show that an atom has a meet-pseudocomplement iff it is coprime.

If a has a meet-pseudocomplement, a is coprime. Otherwise, we would have $a \leqslant b \vee c$, but $a \nleqslant b$ and $a \nleqslant c$.Since a is an atom, $a \wedge b = a \wedge c = O$. This would imply that $b, c \leqslant a_*$, whence $b \vee c \leqslant a_*$ and $O = (b \vee c) \wedge a = a$. Conversely, if a is coprime, let $a_* = \bigvee\{b \in L \mid a \nleqslant b\}$. Since a is coprime, $a \nleqslant a_*$, whence $a \wedge a_* = O$. If $a \wedge b = O$, $a \nleqslant b$ and $b \leqslant a_*$. The proof of (c) is dual. ■

Completely coprime and completely prime elements permit a lattice to be decomposed into two disjoint lattices. The proof of the following theorem is left as an exercise.

THEOREM 6. *Let L be a complete lattice. Then the following are equivalent.*

(i) *L is the disjoint union of A and B where A and B are complete sublattices of L such that A is closed from above and B is closed from below.*

(ii) *L contains a completely coprime element, a.*

(iii) *L contains a completely prime element, b.* ■

The author is indebted to Curtis Greene for suggesting the following corollary.

COROLLARY. *Let L be a complete lattice, C(P) the poset of complete coprimes (primes) in L with the ordering induced by L. Then C and P are isomorphic posets.*

Proof. Let $\alpha: C \to L$ be given by $\alpha(c) = \sup\{y \in L \mid y \ngeqslant c\}$. It is easy to see that $\alpha(c)$ is completely prime for all $c \in C$. Thus, α maps from C into P. Dually, let $\beta: P \to C$ be given by $\beta(p) = \inf\{y \in L \mid y \nleqslant p\}$. It is easy to see that both α and β are isotone. We now show that $\beta \circ \alpha$ is the identity, whence by duality α and β are isomorphisms.

Let $c \in C$. Since c is completely coprime, $\alpha(c) \ngeqslant c$, whence $c \geqslant \beta(\alpha(c))$. As in Theorem 6, c and $\alpha(c)$ partition L. Thus, either $\beta(\alpha(c)) \leqslant \alpha(c)$ or $\beta(\alpha(c)) \geqslant c$. The first case can't happen since $\alpha(c)$ is completely prime, so, $\beta(\alpha(c)) \geqslant c$, whence $\beta(\alpha(c)) = c$. ■

DEFINITION 4. A decomposition of the type described in Theorem 6 is called a *complete coprime/prime decomposition.* The pair (a, b) where $a = \inf A$ and $b = \sup B$ is called the *complete coprime/prime pair associated with the decomposition.* The mappings $f: A \to B$ given by $f(x) = x \wedge b$ and $f: B \to A$ given by $g(x) = a \vee x$ are called the *decomposition maps.* ■

REMARK 3. Theorem 6 shows that if a is a complete coprime then there is a complete prime b such that $L = A \cup B$ where A and B are disjoint sublattices of L and $A = \{x \in L \mid x \geqslant a\}$ while $B = \{x \in L \mid x \leqslant b\}$, and vice versa.

In finite lattices, every coprime/prime pair (a, b) has an associated arc from a to b in $P(L)$. Furthermore, if j is any joint-irreducible such that $j \nleqslant b$, then $j \geqslant a$. Dually, if m is any meet-irreducible such that $m \ngeqslant a$ then $b \geqslant m$. ■

Algebraic topologists have found the concept of a retract useful in understanding the topological structure of objects. This concept makes sense for lattices and provides an alternative way of picturing coprime/prime decompositions.

DEFINITION 5. A lattice K is a *retract* of a lattice L if there are lattice homomorphisms $\alpha: K \to L$ and $\beta: L \to K$ such that $\beta\alpha$ is the identity on K. K is called a *strong retract* of L if $\alpha(K)$ is a subinterval of L. If we require that the maps be isotone rather than lattice homomorphisms we use the terms *order retract* and *strong order retract* (see [23]). ■

The following lemma follows from Schwann's Lemma ([6] p. 73 and Lemma 3, p. 82).

LEMMA 2. *Let L be a complete lattice having a complete coprime/prime decomposition $A \cup B$. Let a and b be the associated pair and let $f: A \to B$ and $g: B \to A$ be the decomposition maps. Then the following are true.*

(i) *f is inf-preserving, g is sup-preserving and both are isotone. Both A and B are order-retracts of L.*

(ii) *For all $y \in A$, $y \geqslant f(y)$ and $g(f(y)) \leqslant y$. For all $x \in B$, $x \leqslant g(x)$ and $f(g(x)) \geqslant x$.*

(iii) *For all $y \in A$, $f(g(f(y))) = f(y)$, and for all $x \in B$, $g(f(g(x))) = g(x)$.*

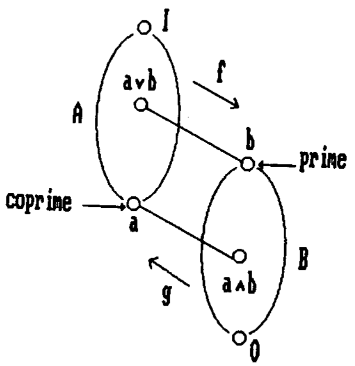


Fig. 2. A schematic diagram of a coprime/prime decomposition.

(iv) *g is injective if and only if for all* $x \in B$, $f(g(x)) = x$ *if and only if f is surjective. If* $f(g(x)) = x$ *for all* $x \in B$, *then a and b form a* modular pair (*see* [6] *p.* 82). *In this case, B is an order-retract of A.*

(v) *f is injective if and only if for all* $y \in A$, $g(f(y)) = y$ *if and only if g is surjective. If* $g(f(y)) = y$ *for all* $y \in A$, *then a and b form a* dual modular pair. *In this case, A is an order-retract of B.*

(vi) *Let* $y \in A$ *and* $x \in B$. *Then* $x \vee y = g(x) \vee y$ *and* $x \vee y = x \wedge f(y)$. ■

Figure 2 is a schematic drawing showing a coprime/prime decomposition and the assocaited decomposition maps. Note that O_A need not cover O_B and I_A need not cover I_B.

Lattices having coprime/prime decompositions can be constructed from arbitrary inf-preserving or sup-preserving maps. Theorem 7 shows how to do this for inf-preserving maps. Stating the dual result is left to the reader. A more general form of this construction is described in [12] so the proof is omitted.

THEOREM 7. *Let* $f: A \to B$ *be an inf-preserving map between complete lattices. Define L to be the disjoint union of A and B. Define a relation* $x \leqslant y$ *on L as follows:* $x \leqslant y$ *iff (a)* $x, y \in A$ *and* $x \leqslant_A y$ *or (b)* $x, y \in B$ *and* $x \leqslant_B y$ *or (c)* $x \in B$ *and* $y \in A$ *and* $f(y) \geqslant_B x$. *Then* $(L, \leqslant)$ *is a complete lattice with sublattices A and B which provide a coprime/prime decomposition of L. The original map f corresponds to the map f of the coprime/prime decomposition.* ■

REMARK 4. The map g of the coprime/prime decomposition in Theorem 7 is the canonical sup-preserving dual of f. For details see ([19] Theorem 2.2). ■

THEOREM 8. (a) *If L has a coprime/prime decomposition then the two element chain,* C_2, *is a strong retract of L. If the two element chain is a retract of a lattice L such that map* $\beta: L \to C_2$ *preserves arbitrary sups and infs, then L has a coprime/prime decomposition.*

(b) *Suppose L is a complete lattice with a coprime/prime decomposition and decomposition maps f, g. Suppose further that f (g) is a surjective lattice homomorphism. Then B (A) is a strong retract of L. If in addition g (f) is also a lattice homomorphism then B (A) is a retract of A (B).*

Proof. (a) If L has a coprime/prime decomposition with coprime/prime pair (a, b) let $\alpha: C_2 \to L$ be given by $\alpha(O) = b$ and $\alpha(I) = a \vee b$ and $\beta: L \to C_2$ be given by $\beta(x) = O$ for $x \in B$ and $\beta(x) = I$ for $x \in A$. It is easy to check that α and β are lattice homomorphisms and that $\beta\alpha$ is the identity on C_2.

Conversely, suppose that C_2 is a retract of L and β preserves arbitrary sups and infs. Let $A = \beta^{-1}(I)$ and $B = \beta^{-1}(O)$. It is easy to see that A and B satisfy the conditions of Theorem 6.

(b) Let (a, b) be the coprime/prime pair associated with A and B. It is clear that B is a subinterval of L and that α can be taken as the inclusion map. Define $\beta: L \to B$ by $\beta(x) = x$ if $x \in B$ and $\beta(x) = f(x)$ if $x \in A$. It is clear that $\beta\alpha$ is the identity on B so we need only show that β is a lattice homomorphism. Let $x, y \in L$ and consider $\beta(x \vee y)$. If $x, y \in B$ then $\beta(x \vee y) = x \vee y = \beta(x) \vee \beta(y)$, If $x, y \in A$ then $\beta(x \vee y) = f(x \vee y) = f(x) \vee f(y) = \beta(x) \vee \beta(y)$ since f is a homomorphism. The only case left to consider is when $x \in B$ and $y \in A$. Here $\beta(x \vee y) = f(x \vee y) = f(x \vee a \vee y) = f(g(x) \vee y)$. Since f is a lattice homomorphism $f(g(x) \vee y) = f(g(x)) \vee f(y) = x \vee f(y) = \beta(x) \vee \beta(y)$ using Lemma 2(iv).

It remains to show that β preserves meets. If $x, y \in A$, then $\beta(x \wedge y) = f(x \wedge y) = f(x) \wedge f(y) = \beta(x) \wedge \beta(y)$. If $x, y \in \beta$, then $\beta(x \wedge y) = x \wedge y = \beta(x) \wedge \beta(y)$. Thus, we may assume that $x \in B$ and $y \in A$. Then $\beta(x \wedge y) = x \wedge y = x \wedge y \wedge b = x \wedge f(y) = \beta(x) \wedge \beta(y)$.

If g is a lattice homomorphism, then from Lemma 2(iv) it is injective and $f(g(x)) = x$ for all $x \in B$. Thus, B is a retract of A. ■

REMARK 5. The one element lattice is a strong retract of every lattice, so in view of Theorem 8 the condition that a lattice has a prime/coprime decomposition is not an overly restrictive condition. We note that in Theorem 8(b) even if f is a lattice homomorphism, g need not be a lattice homomorphism. Figure 3 shows a lattice for which f is a surjective lattice homomorphism but g is not a lattice homomorphism.

The properties of being retracts or strong retracts are transitive in the sense that if A is a (strong) retract of B, and B is a (strong) retract of C, then A is a (strong) retract of C.

Coprime/prime decompositions can be used to prove that a lattice is complemented. The first part of Theorem 9 below appears in ([25] Theorem 15; p. 51), and the remainder is left as an exercise for the reader.

THEOREM 9. *Let L be a complemented lattice. Every coprime in L is an atom (covers O) and every prime is a coatom (covered by I). Further, if* $a, b \in L$ *are a coprime and prime respectively such that* $a \nleqslant b$ *then a and b are complements. In such cases, the complement of any element* $x \leqslant b$ *is* $\geqslant a$ *and vice versa.* ■

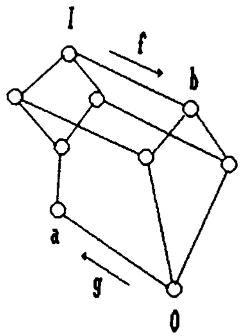


Fig. 3. f is a surjective homomorphism, g is not a homomorphism.

The following theorem provides a mechanism for using coprime/prime decompositions to prove that a lattice is complemented.

THEOREM 10. *Let L be a complete lattice having a complete coprime/prime decomposition $A \cup B$ with associated complete coprime/prime pair (a, b), and let f, g be the decomposition maps. Assume further that a and b are complements. Then if A (B) is complemented and $f\,(g)$ is injective, then L is complemented.*

Proof. Let $y \in A$ and $y' \in A$ be its complement in A. Thus, $y \wedge y' = a$ and $y \vee y' = I$. Let $c = f(y')$. We will show that c is a complement for y in L. Note that $c \wedge y = (b \wedge y') \wedge y = b \wedge (y' \wedge y) = b \wedge a = O$. On the other hand, $c \vee y = c \vee (a \vee y) = (c \vee a) \vee y = g(f(y')) \vee y = y' \vee y = I$ since $g(f(y')) = y'$ by Lemma 2(v).

Let $x \in B$ and let c be the complement in A of $g(x)$. We will show that c is the complement of x in L. $c \vee x = (c \vee a) \vee x = c \vee g(x) = I$. $c \wedge x \leqslant c \wedge g(x) = a$. Also, $c \wedge a \leqslant x \leqslant b$, so $c \wedge x \leqslant a \wedge b = O$. ■

EXAMPLE 2. A simple application of Theorem 10 proves that Bool$(n + 1)$ is complemented assuming that Bool(n) is complemented.

EXAMPLE 3. Theorem 10 is not the best possible theorem. Figure 4(i) shows a lattice L having a coprime/prime decomposition $A \cup B$ where B (dark circles) is complemented but L is not complemented (q has no complement in L). Figure 4(ii) shows a complemented lattice L having a coprime/prime decomposition for which B is complemented but g is not injective. ■

4. Extremal Lattices

This section characterizes extremal lattices in terms of their posets of irreducibles. It shows that no algebraic characterization of this class is possible, so only combinatorial characterizations are possible. The following theorem contains the primary results. *For the remainder of the paper, all lattices are assumed finite.*

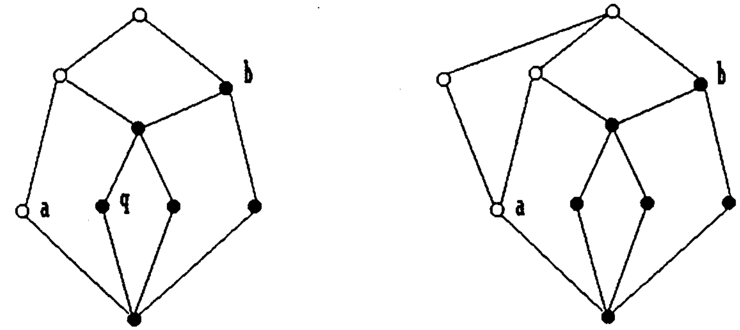


Fig. 4. Two lattices illustrating the limits of Theorem 10.

THEOREM 11. *Let L be a finite lattice and $P(L)$ its poset of irreducibles. Then L is meet-extremal with n meet-irreducibles iff one can number the meet-irreducibles from 1 to n $(m_1, m_2, \ldots, m_n)$ and the join-irreducibles from 1 to p $(p \geqslant n)$ $(j_1, j_2, \ldots, j_p)$ such that for $i = 1 \ldots n$ (j_i, m_i) is an arc in $P(L)$ and if (j_i, m_q) is an arc in $P(L)$, then $q \leqslant i$. We leave stating the dual to the reader.*

Proof. Necessity: Consider a chain, $C = c_0 < c_1 < \cdots < c_n$, of length n in L. Consider the corresponding chain in $\Gamma(L)$. Every element in $\Gamma(L)$ is a subset of $M(L)$ and hence can contain at most n elements. This means that c_i has exactly i elements in its representation in $\Gamma(L)$. Let m_i be the unique element in the representation of c_i that is not in the representation of c_{i-1}. Let j_i be a join-irreducible such that $c_{i-1} \vee j_i = c_i$. First, m_i is in $Ou(j_i)$. Second, if (j_i, m_q) is an arc in $P(L)$ then $q \leqslant i$ since c_i is represented by the set $\{m_1, m_2, \ldots, m_i\}$ in $\Gamma(L)$.

Sufficiency: By Lemma 1 length$(L) \leqslant |M(L)| = n$. Consider the chain in $\Gamma(L)$ formed by $Ou(\{\ \}) \subset Ou(j_1) \subset Ou(\{j_1, j_2\}) \subset \cdots \subset Ou(\{j_1, j_1, \ldots, j_n\})$ which has length n since $Ou(\{j_1, \ldots, j_i\}) = \{m_1, \ldots, m_i\}$. ■

COROLLARY. *Let L be meet-extremal and $\Gamma(L)$ numbered as in Theorem* 11. *Let $j \in J(L)$ and $q = max\{i \mid m_i \in Ou(j)\}$. Then j is coprime iff $\forall j_i \in J(L)$, $m_q \in Ou(j_i)$ implies that $Ou(j) \subseteq Ou(j_i)$.* In this case, *(j, m_q) is a coprime/prime pair. The dual result holds for join-extremal lattices.*

Proof. Sufficiency was discussed in Remark 2 so we need only prove necessity. Suppose that j is coprime and $m_q \in Ou(j_i)$ but $Ou(j) \nsubseteq Ou(j_i)$. It follows that $j \leqslant \sup(\{j_k \mid k < q$ and $m_k \in Ou(j)\} \cup \{j_i\})$, but $j \nleqslant x$ for $x \in \{j_k \mid k < q$ and $m_k \in Ou(j)\} \cup \{j_i\}$, which contradicts the fact that j is coprime.

It remains to show that (j, m_q) is a coprime/prime pair. If j is a coprime, then the corresponding prime is $p = \bigvee\{\alpha \in L \mid \alpha \ngeqslant j\}$. Since L is a finite lattice, $p = \bigvee\{\alpha \in L \mid \alpha \ngeqslant j$ and $\alpha \in J(L)\}$. From before, $\alpha \in J(L)$, $\alpha \ngeqslant j$ iff $m_q \notin Ou(\alpha)$ iff $\alpha \leqslant m_q$. Also, $m_q = \bigvee\{\alpha \in J(L) \mid \alpha \leqslant m_q\}$, so $p = m_q$. ■

THEOREM 12. *If L is a join-extremal lattice then* $\forall x \in L$, $h(x) = |J(x)|$, *where* $h(x)$ *is the height of* x. *If L is a meet-extremal lattice then* $\forall x \in L$, $d(x) = |M(x)|$ *where* $d(x)$ *is the depth of* x.

Proof. The proof for the join-extremal case is as follows. Since $L_2 = \{y \in L \mid y \leqslant x\}$ is a sublattice of L and $J(L_2) = J(x)$, it follows that $h(x) = \text{length}(L_2) \leqslant |J(x)|$. Represent $J(x)$ as $\{j_{f(1)}, j_{f(2)}, \ldots, j_{f(|J(x)|)}\}$ where $f(1) > f(2) > \cdots > f(|J(x)|)$. Theorem 11 shows that $j_{f(1)} < j_{f(1)} \vee j_{f(2)} < \cdots < j_{f(1)} \vee \cdots \vee j_{f(|J(x)|)}$ is a chain of length $|J(x)|$ since $m_{f(i)}$ is in the representation of $j_{f(1)} \vee \cdots \vee j_{f(i)}$ in $\Gamma(L)$ but not in the representation of $j_{f(1)} \vee \cdots \vee j_{f(i-1)}$. Thus, $h(x) = |J(x)|$. ■

COROLLARY 1. *A lattice L is join-extremal iff for all* $x \in L$, $h(x) = |J(x)|$. *Dually, a lattice L is meet-extremal iff for all* $x \in L$, $d(x) = |M(x)|$.

Proof. If L is join-extremal, $h(x) = |J(x)|$ from Theorem 12. On the other hand, if $h(x) = |J(x)|$ we have $\text{length}(L) = h(I) = |J(I)| = |J(L)|$ so L is join-extremal. ■

COROLLARY 2. *If L is join-extremal, then every ideal is a join-extremal lattice. Dually, if L is meet-extremal, then every dual ideal is a meet-extremal lattice.*

Proof. Let M be an ideal of L and let $x = \sup M$. It is easy to see that $J(M) = J(x)$ and that $\text{length}(M) = h(x)$. The result now follows from Corollary 1. ■

THEOREM 13. *A bidigraph (X, Y, Arcs) is P(L) for an extremal lattice L iff*

(i) $|X| = |Y| = n$, *and*

(ii) *X and Y can be numbered from* 1 *to n such that* $(x_i, y_i) \in$ *Arcs for all i and if* $(x_i, y_j) \in$ *Arcs then* $i \geqslant j$.

or

(ii)′ *X and Y can be numbered from* 1 *to n such that* $(x_i, y_i) \in$ *Arcs for all i and if* $(x_i, y_j) \in$ *Arcs then* $i \leqslant j$.

Proof. The proof for (i) and (ii) runs as follows. Note that a numbering satisfying (ii) can be transformed into a numbering satisfying (ii)′ simply by replacing the number i by $n + 1 - i$.

Necessity follows from Theorem 11. Now suppose a bidigraph $D = (X, Y, \text{Arcs})$ satisfies conditions (i) and (ii). Theorem 2 and condition (ii) show that D is $P(L)$ for some L, since if $Ou(W) = Ou(x_i)$ then no $x_r \in W$ where $r > i$. On the other hand, $y_i \in Ou(x_i)$ but $y_i \notin Ou(x_j)$ for all $j < i$, so $x_i \in W$. The argument for the y_i is dual. Since we know that $D = P(L)$ for some lattice L, it follows from Theorem 11 that L is extremal. ■

The following definition was suggested by Garrett Birkhoff. It focuses attention on the key role that chains play in the analysis of p-extremal lattices.

DEFINITION 6. Let L be a lattice of finite length. A chain $C = c_0 < c_1 < \cdots < c_k$ in L is called *join-irredundant* if $|J(c_{i+1}) - J(c_i)| = 1$ for $i = 0 \cdots k-1$. C is called *meet-irredundant* if $|M(c_i) - M(c_{i+1})| = 1$ for $i = 0 \cdots k-1$. C is called *irredundant* if it is both join-irredundant and meet-irredundant. ■

REMARK 6. The preceding results show that join-extremal lattices having join-rank n have a join-irredundant chain of length n. Similarly meet-extremal lattices having meet-rank n have a meet-irredundant chain on length n. Finally, extremal lattices having a join-rank of n have an irredundant chain of length n. In particular, all chains having length n are (join-, meet-) irredundant as appropriate.

The numbering schemes discussed in Theorems 11 and 13 make is easy to construct (join-, meet-) irredundant chains. For example, if $j_1, j_2, \ldots, j_n$ are the join-irreducibles of a join-extremal lattice of length n, then $j_1 < j_1 \vee j_2 < \cdots < j_1 \vee j_2 \vee \cdots \vee j_n$ is a join-irredundant chain of length n. We will use this observation to construct maximal irredundant chains in Tamari lattices. ■

THEOREM 14. (i) *Any finite lattice K can be embedded as a sublattice in a meet-extremal (join-extremal) lattice L so it corresponds to B(A) in a coprime/prime decomposition of L.*

(ii) *Any finite lattice L is isomorphic to an interval of a finite extremal lattice.*

Proof. (i) Suppose that K has n meet-irreducibles numbered $m_1, m_2, \ldots, m_n$. Let $D = (X, Y, \text{Arcs})$ be a bidigraph defined as follows. $X = J(K) \cup \{p_0, p_1, \ldots, p_n\}$, $Y = M(K) \cup \{q_0\}$ where the p_i's and q_0 are chosen so they are distinct from all elements of K and from each other. $\text{Arcs}(D) = \text{Arcs}(P(K)) \cup \{(p_i, q_0) \mid i = 0, \ldots, n\} \cup \{(p_i, m_i) \mid i = 1, \ldots, n\}$. It follows from Theorem 2 that D is the poset of irreducibles of some lattice L. Furthermore, using the ordering $q_0, m_1, \ldots, m_n$ and $p_0, p_1, \ldots, p_n, J(K)$ it follows from Theorem 11 that L is meet-extremal. Note that $Ou(p_0) = \{q_0\}$ in D, so that p_0 is an atom in L. p_0 is coprime in L since if $Ou(p_0) \subseteq Ou(W)$, for some $w \in W$, $q_0 \in Ou(w)$, so $p_0 \leqslant w$. The elements of $\Gamma(L)$ corresponding to B in the coprime/prime decomposition induced by p_0 correspond to $\{Ou(W) \mid W \subseteq J(K)\}$, so B is isomorphic to K.

(ii) The construction in this case is an extension of the construction used in (i). Suppose that K has k join-irreducibles $\{j_1, \ldots, j_k\}$ and n meet-irreducibles $\{m_1, \ldots, m_n\}$. Let $D = (X, Y, \text{Arcs})$ be a bidigraph defined as follows. $X = J(K) \cup \{p_0, \ldots, p_n\} \cup \{r_0\}$, $Y = M(K) \cup \{q_0\} \cup \{s_0, \ldots, s_k\}$ and $\text{Arcs} = \text{Arcs}(P(K)) \cup \{(p_i, q_0) \mid i = 0, \ldots, n\} \cup \{(p_i, m_i) \mid i = 1, \ldots, n\} \cup \{(r_0, s_i) \mid i = 0, \ldots, k\} \cup \{(j_i, s_i) \mid i = 1, \ldots, k\}$. Note that $|X| = |Y| = k + n + 2$. It is easy to see that D satisfies Theorem 13 so $D = P(L)$ for some extremal lattice L. The reader can verify that K is isomorphic to the sublattice $\{Ou(W) \mid W \subseteq J(K) \cup \{r_0\}$ where $r_0 \in W\}$ which is isomorphic to the interval $[r_0, r_0 \vee j_1 \vee \cdots \vee j_k]$. ■

COROLLARY. *Extremal lattices cannot be characterized algebraically.* ■

EXAMPLE 4. We will show how to embed the modular lattice M_5 into an extremal lattice. Figure 5(i) shows $P(M_5)$ while Figure 5(ii) shows the bidigraph

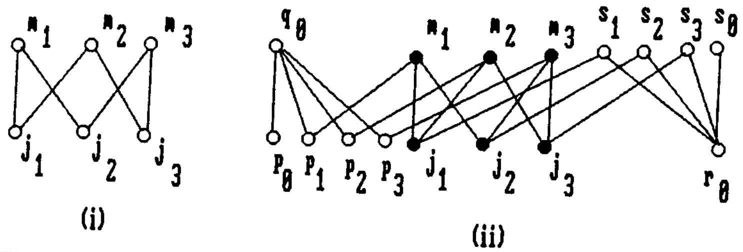


Fig. 5. Embedding M_5 into an extremal lattice.

constructed following the recipe in Theorem 14(ii). The elements p_i $(i \geqslant 1)$ match up with the original meet-irreducibles and connect to these as well as q_0. p_0 is needed to ensure that the constructed bidigraph is the poset of irreducibles of a lattice. Similarly, the s_i $(i \geqslant 1)$ correspond to the original join-irreducibles while s_0 and r_0 are needed to satisfy the requirements of Theorem 2. The lattice corresponding to the new poset of irreducibles is shown in Figure 6. It has 39 elements. The sublattice isomorphic to M_5 is indicated by the shaded circles. The lattice in Figure 6 has a horizontal axis of symmetry. Also an irredundant chain of length 8 is indicated by the thicker edges. ■

Extremal lattices have very handy coprime/prime decompositions. The key result here is Theorem 15 which shows that p-extremal lattices have coprime/prime decompositions in which one of A or B is of the same type as the original lattice. For extremal lattices one part of the decomposition is extremal while the other is join-extremal or meet-extremal depending on the type of decomposition chosen.

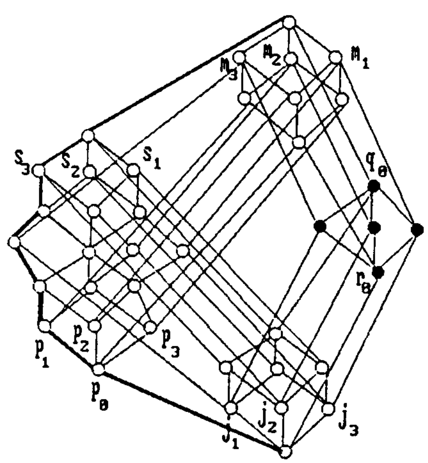


Fig. 6. M_5 embedded into an extremal lattice.

THEOREM 15. *Let L be a meet-extremal lattice. L has an atom a which is coprime. Furthermore*:

1. *the decomposition map $g: B \to A$ is injective and $g(x)$ covers x;*
2. *A is meet-extremal and contains all but 1 of the meet-irreducibles of L and length(A) = length$(L) - 1$;*
3. *the remaining meet-irreducible of L is the prime b corresponding to the coprime a;*
4. *every join-irreducible of A considered as a lattice is either a join-irreducible of L or $a \vee j$ where j is a join-irreducible of L that is contained in B.*
5. *A and B are order retracts of L and B is an order-retract of A.*
6. *If L is extremal, A is extremal and B is join-extremal.*

We leave formulating the dual statements to the reader.

Proof. By Theorem 11, $J(L)$ contains j_1 such that $Ou(j_1) = \{m_1\}$, so j_1 is a coprime atom in L. In our decomposition $a = j_1$ and $b = m_1$. The set representing b in $\Gamma(L)$ is $Ou(J(b))$. It does not include b since there are no arcs from elements of $J(b)$ to b. The map $g: B \to A$ is given by $g(x) = x \vee a$. In $\Gamma(L)$, $g(x)$ adds m_1 to subsets of $M(L)$. It is clear that g is an injection and that $g(x)$ covers x.

A is meet-extremal by Corollary 2 of Theorem 12. As noted in the preceding paragraph, $m_1 = b$ is in B. For $i = 2 \cdots n$ where L has n meet-irreducibles, there is no arc from j_1 to m_i so $j_1 \in J(m_i)$, which means that m_1 is in the set representing m_i in $\Gamma(L)$. Thus for $i > 1$, $m_i \in A$. Since A contains 1 less meet-irreducible than L and is meet-extremal, length(A) = length$(L) - 1$.

Suppose q is join-irreducible in A but not in L. Then q must cover some element w in B. Suppose w is not join-irreducible, then there exist $r, s \in B$ such that $r \vee s = w$. In $\Gamma(L)$ join corresponds to union and it is easy to see that $g(r) \vee g(s) = g(w) = q$. Since g is injective, q is not join-irreducible in A since $g(r), g(s) \in A$. Thus w must be join-irreducible in L. The order retract properties follow from Lemma 2.

Finally, if L is extremal we will show that A is extremal and that B is join-extremal. That B is join-extremal follows immediately from Corollary 2 of Theorem 12 since L is join-extremal. We know that A is meet-extremal so we need only show that it is join-extremal.

Suppose L has n join-irreducibles and n meet-irreducibles. We know that length$(A) = n - 1$, so from Lemma 1 A must contain at least $n - 1$ join-irreducibles. Each join-irreducible of A is either a join-irreducible of L or $a \vee j$ where j is a join-irreducible of L contained in B. Thus A can have at most $n - 1$ join-irreducibles since a is a join-irreducible of L that cannot contribute a join-irreducible to A. Thus A contains exactly $n - 1$ join-irreducibles and is extremal. The order retract properties follow from Lemma 2. ■

5. Distributive Lattices

The numbering of Theorem 13 can be used to characterize the poset of irreducibles of distributive lattices. This characterization leads to a new proof of the fact that a finite lattice is distributive iff it is an extremal lattice satisfying the Jordan–Dedekind chain condition. This characterization also provides additional information about coprime/prime decompositions in the case when L is distributive.

THEOREM 16. *Let L be an extremal lattice. Assume that the join-irreducibles and meet-irreducibles are numbered as in Theorem 13(ii). Then L is distributive iff for all i and q, $m_q \in Ou(j_i)$ implies $Ou(j_q) \subseteq Ou(j_i)$.*

Proof. By Theorem 5(a), L is distributive iff all the join-irreducibles are coprime. The Corollary to Theorem 11 shows that the condition stated above is equivalent to the join-irreducibles being coprime. ■

REMARK 7. Theorem 16 provides a much simpler characterization of finite distributive lattices than Theorem 11 of [19]. The proof of Theorem 17 gives an alternative proof of the combinatorial characterization of distributive lattices ([1] Theorem 4.6, [20] Theorem 3.1).

THEOREM 17. *An extremal lattice, L, satisfying the Jordan–Dedekind chain condition is distributive.*

Proof. Start with the numbering of Theorem 13(ii). Suppose $m_q \in Ou(j_i)$ but $Ou(j_q) \not\subseteq Ou(j_i)$. The set $e = \{m_1, m_2, \ldots, m_i\}$ is in $\Gamma(L)$ and $O < j_1 < j_1 \vee j_2 < \cdots < j_1 \vee \cdots \vee j_i$ is a maximal chain of length i from O to e. We can construct a shorter maximal chain as follows. First construct a maximal chain from O to j_i. This can be constructed using only join-irreducibles numbered $\leqslant i$ and will not include j_q since $j_q \not\leqslant j_i$. Now complete the chain from j_i to e by first using any needed elements from $j_1, \ldots, j_{q-1}$ that were not used to create the first maximal chain, and then using any needed elements from $j_{q+1}, \ldots, j_{i-1}$. This produces a maximal chain from O to e having at most $i-1$ steps which contradicts the fact that the lattice satisfies the Jordan–Dedekind chain condition. Thus, $j_q \leqslant j_i$ and the result follows from Theorem 16. ■

THEOREM 18. *Let L be a distributive lattice and (A, B) a coprime/prime decomposition of the type described in Theorem 15. Then both A and B are distributive (and hence extremal), while f and g are both lattice homomorphisms, B is isomorphic to a sublattice of A, B is a strong retract of L, and B is a retract of A. A dual result holds if (A, B) is the dual of the representation in Theorem 15.*

Proof. A and B are distributive since they are sublattices of distributive lattices. Since L is distributive $f(x) = x \wedge b$ preserves joins as well as meets, and $g(y) = y \vee a$ preserves meet as well as joins. Since g is injective, B is isomorphic to a sublattice of A and by Theorem 8(b) B is a strong retract of L since f is surjective. ■

REMARK 8. Theorem 18 shows that distributive lattices have a decomposition into two distributive lattices such that one is a sublattice of the other. Furthermore, one of the lattices is a strong retract of the original lattices. Applying this result

inductively leads to a sequence of strong retracts that ends with the one element lattice. ■

6. Locally Distributive Lattices

DEFINITION 7. A lattice L is *lower locally distributive* if for each $x \in L$ the interval $[x_*, x]$ is a Boolean algebra where $x_* = \inf\{y \in L \mid x \text{ covers } y\}$. L is *upper locally distributive* if for each $x \in L$ the interval $[x, x^*]$ is a Boolean algebra where $x^* = \sup\{y \in L \mid y \text{ covers } x\}$. ■

REMARK 9. For additional information about locally distributive lattices see [1, 7, 16, 24]. In [24] lower locally distributive lattices are called *meet-distributive*. From the definitions of finite lower (upper) locally distributive lattices it follows that they are lower (upper) semimodular and hence by Theorem 14 of ([6] p. 40) they satisfy the Jordan–Dedekind chain condition.

Avann [1] and later Greene and Markowsky [14] show that finite lower (upper) locally distributive lattices are exactly the class of join-extremal (meet-extremal) lattices that satisfy the Jordan–Dedekind chain condition.

Theorem 19 follows from the fact that locally distributive lattices are p-extremal and Jordan–Dedekind. It is a special case of the more general result ([11] Lemma 1) that atoms in semidistributive lattices are coprime. See ([22] p. 194) for a discussion of upper locally distributive lattices and why they are semidistributive.

THEOREM 19. *In finite upper (lower) locally distributive lattices, atoms (coatoms) are coprime (prime).* ■

Theorem 20 characterizes the posets of irreducibles of Jordan–Dedekind, meet-extremal lattices. This characterization provides an alternative proof of the fact that every meet-extremal lattice that satisfies the Jordan–Dedekind chain condition is upper locally distributive. It also proves that both A and B in the decomposition of Theorem 15 are upper locally distributive. An alternative representation of locally distributive lattices is given in [9].

THEOREM 20. *Let L be a finite lattice and $\Gamma(L)$ be the representation derived from the poset of irreducibles. Furthermore, for $x \in L$, let $\Gamma(x)$ be the corresponding element in $\Gamma(L)$. Then the following are equivalent.*

a. *L is meet-extremal and satisfies the Jordan–Dedekind chain condition.*

b. *$\forall x \in L$, if x covers y then $|\Gamma(x) - \Gamma(y)| = 1$.*

Proof. (a) $\to$ (b) Suppose L satisfies the Jordan–Dedekind chain condition and $n = \mathrm{mr}(L) = \mathrm{height}(L)$. It is clear that any maximal chain from O to I must have the form $O = c_0 < \cdots < c_n = I$. Since the lattice is Jordan–Dedekind, if x covers y, there is a maximal chain of length n from O to I that contains both x and y. Since $\mathrm{mr}(L) = n$, (b) follows.

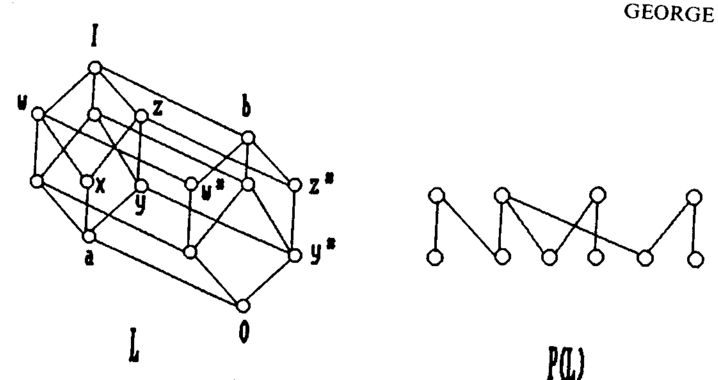


Fig. 7. B is not a sublattice of A and neither f nor g is a homomorphism.

(b) → (a) It is easy to see that a lattice satisfying (b) is Jordan–Dedekind since any maximal chain from y to x must contain exactly $|\Gamma(x) - \Gamma(y)|$ elements. It is meet-extremal because any maximal chain from O to I must contain exactly $|\Gamma(I) - \Gamma(O)| = |\Gamma(I)| = |M(L)|$ elements. ∎

COROLLARY 1. *A meet-extremal lattice that satisfies the Jordan–Dedekind chain condition is upper locally distributive.*

Proof. From Theorem 20 it follows that if x covers y, $\Gamma(x) - \Gamma(y)$ is a singleton. Let y^* be the join of all elements that cover y. It is clear that the interval $[y, y^*]$ is a Boolean algebra since every element in the interval is formed by taking the union of $\Gamma(y)$ and the singletons $\Gamma(x) - \Gamma(y)$ where x ranges over all elements that cover y. ∎

COROLLARY 2. *Let L be a finite, upper locally distributive lattice. Decompose L along the lines of Theorem 15. Then both A and B are upper locally distributive.* ∎

REMARK 10. Corollary 2 also follows from the fact that semidistributivity and semimodularity are inherited by sublattices.

The nice sublattice and mapping properties of distributive lattices do not hold for locally distributive lattices. Figure 7 shows an upper locally distributive lattice such that B is not a sublattice of A and neither f nor g is a lattice homomorphism. Also shown is the poset of irreducibles of the lattice.

7. Tamari Associativity Lattices

Given a binary operation $*$ and $n+1$ appropriate elements there are many ways to parenthesize the expressions $x_1 * x_2 * \cdots * x_{n+1}$. The collection of parenthesized expressions can be made into a lattice based on replacing products of the form

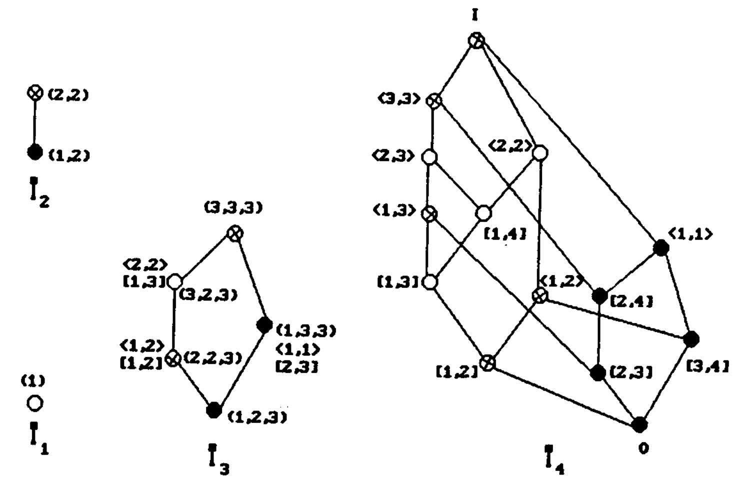


Fig. 8. T_1, T_2, T_3 and T_4.

$(a * b) * c$ by $a * (b * c)$. Bennett and Birkhoff [4] call the corresponding lattices the *Tamari lattices* T_n and discuss their history and some of their properties. We will prove that the Tamari lattices are extremal and complemented. Furthermore, they share the nice decomposition properties of distributive lattices and Boolean algebras. Figure 8 shows T_1, T_2, T_3 and T_4 so as to highlight the decompositions.

We will use the alternative definition (Definition 8) of Tamari lattices discovered by Huang and Tamari [15] and also discussed in [26] and [4].

DEFINITION 8. For each n let T_n be the set of all n-vectors $(v_1, v_2, \ldots, v_n)$ of positive integers $\leqslant n$ satisfying the following properties.

(i) $i \leqslant v_i$ for all $i = 1, \ldots, n$

(ii) $i \leqslant j \leqslant v_i$ implies $v_j \leqslant v_i$

T_n is ordered by componentwise comparison. ∎

REMARK 11. Huang and Tamari observed that T_n is closed under pointwise meets in the Cartesian product of n copies of $1 < 2 < \cdots < n$, whence T_n is a lattice. Less obvious is the self-duality of T_n. ([26] Lemmas 8 and 9) and [4] determine the irreducibles of the Tamari lattices. Their results are summarized in Theorem 21. For more information on the Tamari lattices see ([13] pp. 14–15; p. 51, [26] and [4]). [26] also shows that the T_n are splitting lattices in the sense of McKenzie, and the methods in [26] can be used to prove that the T_n are semidistributive, which implies that they are pseudocomplemented.

THEOREM 21. *The join-irreducibles of T_n are exactly the vectors $[j, k]$ for $j < k \leqslant n$ $[j,k]_i = i$ if $i \neq j$ and $[j,k]_j = k$ ($[j,k] = (1, 2, \ldots, j-1, k, j+1, \ldots, n)$). The meet-irreducibles of T_n are exactly the vectors $\langle j, k\rangle$ for $j \leqslant k < n$ where $\langle j, k\rangle_i = n$*

if $i<j$ *or* $i>k$ *and* $\langle j,k\rangle_i = k$ *if* $j \leqslant i \leqslant k$ $(\langle j,k\rangle = (n, n, \ldots, n, k, \ldots, k, n, \ldots, n))$. ■

COROLLARY 1. $([j,k], \langle p,q\rangle) \in Arcs(P(T_n))$ *iff* $p \leqslant j \leqslant q < k$. ■

COROLLARY 2. *In* T_n, O *is the meet of coatoms and* I *is the join of atoms.*

Proof. Consider an element of the form $v = \langle p,p\rangle$ where $p<n$. This is a coatom since if $w > v$, $w_i = n$ for $i \neq p$ and $w_p = q > p$. It follows that $w_p \geqslant w_q = n$, so $w = I$. It is easy to see that $\bigwedge\{\langle p,p\rangle \mid p = 1 \cdots n-1\} = (1, 2, \ldots, n) = O$. Since T_n is self-dual, I is the join of atoms. The atoms are the elements $[j, j+1]$. ■

COROLLARY 3. *The primes of* T_n *are exactly its coatoms and its coprimes are exactly its atoms.*

Proof. First, from Corollary 2 the coatoms are exactly the elements of the form $\langle p,p\rangle$. To see that $\langle p,p\rangle$ is prime, suppose $\inf X \leqslant \langle p,p\rangle$. Since inf is pointwise meet $\exists x \in X$ such that $x_p = p$. It is clear that $x \leqslant \langle p,p\rangle$.

No meet-irreducible of the form $\langle p,q\rangle$ with $p<q$ is prime since O is the meet of coatoms, and no coatom is $\leqslant \langle p,q\rangle$ with $p<q$. ■

THEOREM 22. T_n *is a complemented, join- and meet-pseudocomplemented, extremal lattice of length* $n(n-1)/2$. T_n *has a coprime/prime decomposition such that* B *is isomorphic to* T_{n-1} *and another one such that* A *is isomorphic to* T_{n-1}. *In both these decompositions both* A *and* B *are extremal lattices.*

Proof. The number of join-irreducibles of T_n is equal to the number of pairs (j,k) such that $1 \leqslant j < k \leqslant n$, which is simply the number of pairs taken from n elements without repetition. This number is $n(n-1)/2$. The number of meet-irreducibles is equal to the number of pairs $1 \leqslant p \leqslant q < n$, which is the number of pairs with repetition taken from $1, \ldots, n-1$. This also works out to be $n(n-1)/2$.

Now totally order the join-irreducibles reverse lexicographically so that $[j,k]$ precedes $[r,s]$ if $k<s$, or if $j<r$ and $k=s$. This means that $[j,k]$ comes in position $(k-1)(k-2)/2+j$ in the total order.

Next, totally order the meet-irreducibles using the same reverse lexicographical order, i.e., $\langle j,k\rangle$ precedes $\langle r,s\rangle$ if $k<s$ or if $j<r$ and $k=s$. This means that $\langle j,k\rangle$ comes in position $k(k-1)/2+j$ in the total order.

Corollary 1 to Theorem 21 implies that $([j,k], \langle j, k-1\rangle \in \mathrm{Arcs}(P(T_n))$ and that if $([j,k], \langle p,q\rangle) \in \mathrm{Arcs}(P(T_n))$ then $\langle p,q\rangle$ precedes $\langle j,k-1\rangle$ in the ordering just described. Note that $[j,k]$ comes in position $(k-1)(k-2)/2+j$ as does $\langle j,k-1\rangle$. From Theorem 13 it follows that T_n is extremal and that consequently it has length $n(n-1)/2$.

We will prove that T_n is complemented by induction on n. T_1 is the one element lattice and T_2 is a two element chain. Both are complemented.

Since $Ou([1,2]) = \{\langle 1,1\rangle\}$ in all $P(T_n)$, $[1,2]$ is a coprime. Consider the coprime/prime decomposition generated by the pair $([1,2], \langle 1,1\rangle)$. From Theorem 15 we know A is extremal and B is join-extremal and the map $f\colon B \to A$ is injective. If we can prove that $[1,2]$ and $\langle 1,1\rangle$ are complements and that B is complemented, it will follow from Theorem 10 that T_n is complemented. Pseudo-complementation follows from Theorem 5 and Corollary 3 of Theorem 21.

Now $[1,2] = (2,2,3,4,\ldots,n)$ while $\langle 1,1\rangle = (1,n,n,\ldots,n)$. It is easy to see that $[1,2] \wedge \langle 1,1\rangle = (1,2,3,\ldots,n) = O$. From the proof of Corollary 2 of Theorem 21 we know that $\langle 1,1\rangle$ is a coatom. Since $[1,2] \nleqslant \langle 1,1\rangle$, $[1,2] \vee \langle 1,1\rangle = I$.

We will prove that B is complemented by showing that B is isomorphic to T_{n-1}. For Theorem 15 and Lemma 2 the map $f\colon A \to B$ given by $f(y) = y \wedge \langle 1,1\rangle$ is surjective. This means that $B = \{v \in T_n \mid v_1 = 1\}$. It is easy to see that B is isomorphic to T_{n-1} by considering the map $\Theta\colon B \to T_{n-1}$ given by $\Theta((1 = v_1, v_2, v_3, \ldots, v_n = n)) = (v_2 - 1, v_3 - 1, \ldots, v_n - 1)$. Thus, B is extremal also.

The Tamari lattice is self-dual so it is possible to find a coprime/prime decomposition such that A is isomorphic to T_{n-1}. ■

REMARK 12. The fact that T_n is complemented is attributed by ([13] p. 51; Exercises 30 & 31) to H. Lakser. For a thorough discussion of finite lattices that are both complemented and pseudocomplemented see [8].

The elements of T_n for which $v_1 = 1$ are not the only sublattice of T_n which is isomorphic to T_{n-1}. Another sublattice of T_n isomorphic to T_{n-1} is the set of elements for which $v_j \leqslant n-1$ for all $j \leqslant n-1$. In this last case, however, the complement of this set is not a lattice so we do not get a prime/coprime decomposition of T_n. Another prime/coprime decomposition of T_n is given by the pair $(\langle n-1, n-1\rangle, [n-1, n])$. In this case A is isomorphic to T_{n-1}.

Because T_n is extremal its longest chain has length $n(n-1)/2$. We will now show that the shortest maximal chain in T_n has length $n-1$. ■

LEMMA 3. b *covers* a *in* T_n *if and only if there exists* j *such that* $a_i = b_i$ *for all* $i \neq j$, $a_j < b_j$ *and* $b_j = b_k = a_k$ *where* $k = a_j + 1$.

Proof. Sufficiency: Suppose that $a < c < b$. This can only happen if $a_j < c_j < b_j$. Note that $j \leqslant a_j < k = a_j + 1 \leqslant c_j$, so $b_k = c_k \leqslant c_j < b_j = b_k$ which is impossible.

Necessity: If b covers a let j be the smallest index where a and b differ. Thus, $a_j < b_j$. Let $k = a_j + 1$. Let c be the vector defined by $c_i = a_i$ if $j \leqslant i \leqslant (k-1)$ and $c_i = b_i$ otherwise. We first show that $c \in T_n$. Since $c_i \geqslant a_i \geqslant i$ for all i, the first condition is satisfied. Now suppose that $i \leqslant p \leqslant c_i$. If $i<j$ or $i \geqslant k$, then $c_i = b_i$. Since $b \in T_n$, $c_p \leqslant b_p \leqslant b_i$. If $j \leqslant i \leqslant (k-1)$, then $c_i = a_i$. Since $a \in T_n$, for all i such that $j \leqslant i \leqslant (k-1) = a_j$, $a_i \leqslant a_j$. If $j \leqslant i \leqslant (k-1)$ and $i \leqslant p \leqslant c_i = a_i$, then $p \leqslant a_j = (k-1)$. Thus, $c_p = a_p \leqslant a_i$. It is clear that $a \leqslant c < b$ since $c_j < b_j$. Since b covers a, $a = c$. This means that a and b are identical for $i<j$ and $i \geqslant k$.

Let d be the vector defined by $d_i = a_i$ if $j+1 \leqslant i \leqslant (k-1)$, $d_j = b_k$ and $d_i = b_i$ otherwise. We will show that $d \in T_n$. As before $d_i \geqslant i$. Now let $i \leqslant p \leqslant d_i$. First, suppose that $i<j$ or $i \geqslant k$. If further, $p \neq j$, then $d_p \leqslant b_p \leqslant b_i = d_i$. Now consider

the case $p = j$. If $i \leqslant j \leqslant d_i = b_i$, then $b_j \leqslant b_i$. Since $a_j < b_j$, $j \leqslant k = a_j + 1 \leqslant b_j$ so $b_k \leqslant b_j$. Thus, $d_j = b_k \leqslant b_j \leqslant b_i = d_i$. If $i \leqslant p \leqslant d_i$ and $j < i \leqslant (k-1) = a_j$, then $d_i = a_i$. Since $a \in T_n$, $a_i \leqslant a_j$. It follows that $i \leqslant p \leqslant a_i \leqslant a_j = (k-1)$, so $d_p = a_p \leqslant a_i = d_i$. This leaves only the case $j \leqslant p \leqslant d_j = b_k$. If $p \geqslant k$, then $d_p = b_p \leqslant b_k$, since $b \in T_n$. If $p \leqslant (k-1)$, then $d_p = a_p \leqslant a_j = (k-1) < k \leqslant b_k = d_j$.

Now $a < d$, since $a_j < d_j$. Furthermore, $j \leqslant k = a_j + 1 \leqslant b_j$, so $b_k \leqslant b_j$, whence $d \leqslant b$. Since b covers a, $d = b$. This means that a and b are identical also for i in the range $j+1 \cdots (k-1)$. They differ only at j where $b_j = b_k = a_k$. Note also that $a_{k-1} = a_j$. ■

EXAMPLE 5. Consider the elements $\langle 2, 3\rangle$ $(=(4, 3, 3, 4))$, $\langle 1, 3\rangle$ $(=(3, 3, 3, 4))$, and $[2, 3]$ $(=(1, 3, 3, 4))$ from T_4, which is pictured in Figure 8. $\langle 2, 3\rangle$ does not cover $[2, 3]$ even though they differ in only one component because $1 + [2, 3]_1 = 2$, but $\langle 2, 3\rangle_1 = 4 \neq 3 = \langle 2, 3\rangle_2 = [2, 3]_2$. The reader should check that Lemma 3 implies that $\langle 2, 3\rangle$ covers $\langle 1, 3\rangle$ and that $\langle 1, 3\rangle$ covers $[2, 3]$. ■

THEOREM 23. *The shortest maximal chain in T_n has length $n-1$.*

Proof. The proof is by induction of n with the result being obviously true for $n = 1$ and 2. The prime/coprime decomposition $(\langle 1, 1\rangle, [1, 2])$ used in Theorem 22 shows that T_{n-1} is the interval $[0, \langle 1, 1\rangle]$ in T_n. From the induction hypothesis it follows that there is a maximal chain of length $1 + (n-2)$ in T_n ($n-2$ steps in $[0, \langle 1, 1\rangle]$ and one step from $\langle 1, 1\rangle$ to I). We need only show that there is no shorter chain.

Consider the map $f\colon T_n \to B$ given by $f(y) = y \wedge \langle 1, 1\rangle$ (this is just an extension of the coprime/prime decomposition map to all of T_n). From Lemma 3 it follows that if y covers x then either $f(y) = f(x)$ or $f(y)$ covers $f(x)$ depending on whether x and y differ in the first coordinate. Now consider a maximal chain $C = x_0 < x_1 < \cdots < x_q = I$ of length q in T_n. The chain $f(C)$ is a maximal chain in T_{n-1} of length at most $q-1$ since $f(x_t) = x_{t-1} = f(x_{t-1})$ where t is the smallest index for which $x_t \in A$. Thus $q - 1 \geqslant n - 2$, whence $q \geqslant n-1$. ■

REMARK 13. It was noted earlier that an irredundant chain in an extremal lattice can be produced by forming the sequence $0 < j_1 < j_1 \vee j_2 < \cdots < I$. Using the numbering of Theorem 22 this yields the chain $O = 12345\ldots, 22345\ldots, 32345\ldots, 42345\ldots, \ldots, n2345\ldots, n3345\ldots, n4345, \ldots, \ldots, nnnn\ldots = I$.

We now wish to analyze in greater detail the relationship between T_n and T_{n-1}. To this end we need a better understanding of how joins are computed in T_n. This is given by Lemma 4. ■

LEMMA 4. *The join of x and y in T_n can be computed as follows. First compute w such that $w_i = \max\{x_i, y_i\}$ for all i. Next define z recursively as follows. First, let $z_n = n$. Next, assume that $z_{i+1}, \ldots, z_n$ are defined. New set z_i to $\max\{w_i\} \cup \{z_j \mid j = i+1, \ldots, w_i\}$. Then $z = x \vee y$.*

Proof. In general, w is not a member of T_n, so we must show that z is a member of T_n. For all i, $z_i \geqslant w_i \geqslant x_i \geqslant i$ so z satisfies condition (i) of Definition 8. It remains to show that z satisfies condition (ii) of Definition 8.

The result holds for $i = n$. Assume that the result is true for $i+1, \ldots, n$. We will show that it holds for i. Now suppose that $i \leqslant j \leqslant z_i$. If $i = j$, condition (ii) holds trivially. Thus, we may assume that $i < j \leqslant z_i$. If $j \leqslant w_i$ then by definition of $z_i, z_i \geqslant z_j$. Thus, we may assume that $w_i < j$. This implies that $z_i > w_i$ so there must exist a k such that $z_i = z_k$ and $i+1 \leqslant k \leqslant w_i$. Now $i + 1 \leqslant k \leqslant j \leqslant z_k = z_i$, whence by the recursive hypothesis $z_j \leqslant z_k = z_i$. This shows that $z \in T_n$.

It is clear that z is an upper bound of $\{x, y\}$ so it only remains to prove that it is the least upper bound of $\{x, y\}$. Let u be any upper bound of $\{x, y\}$. It is clear that $u \geqslant w$. We will now prove by induction that $u_i \geqslant z_i$.

The result is trivially true for $i = n$. Assume it is true for $i+1, \ldots, n$ and consider the situation for u_i. Since $u \in T_n$, $u_i \geqslant u_j$ for all $i \leqslant j \leqslant u_i$. Since $u_i \geqslant w_i$, $u_i \geqslant u_j$ for all $i \leqslant j \leqslant w_i$, $u_i \geqslant \max\{w_i, u_{i+1}, \ldots, u_q\} \geqslant \max\{w_i, z_{i+1}, \ldots, z_q\} = z_i$, where $q = w_i$. Thus, $z = x \vee y$. ■

REMARK 14. Theorem 24 refines the earlier results and gives us more insight into the relationship between T_n and T_{n-1}. Formulating the dual of Theorem 24 is left to the reader. ■

THEOREM 24. *Decompose T_n into A and B as in Theorem 22 so B is isomorphic to T_{n-1}. The decomposition maps f and g are lattice homomorphisms, B is a sublattice of A and T_{n-1} is a strong retract of T_n. B is a retract of A.*

Proof. The coprime/prime pair for this decomposition is $([1, 2], \langle 1, 1\rangle)$. The map f is given by $f(x) = x \wedge \langle 1, 1\rangle$. We know that f preserves meets so we need only check that it preserves joins. Since $\langle 1, 1\rangle = (1, n, n, \ldots, n)$, $f(x) = (1, x_2, x_3, \ldots, x_n)$. Let $x, y \in A$ and $z = x \vee y$. If you examine the algorithm given in Lemma 4 for computing $x \vee y$, you will see that for $i = 2, \ldots, n$, $z_i = t_i$ where $t = f(x) \vee f(y)$ since none of these computations depend on the first coordinate. Thus, f is join-preserving and hence a homomorphism.

From Lemma 4, $g(1, x_2, \ldots, x_n) = [1, 2] \vee (1, x_2, \ldots, x_n) = (x_2, x_2, \ldots, x_n)$. We know that g preserves joins; it is straightforward to verify that it preserves meets as well. Thus, $B(T_{n-1})$ is a sublattice of A. From Theorem 8(b) it follows that T_{n-1} is a strong retract of T_n and that B is a retract of A. ■

REMARK 15. Figure 8 illustrates Theorem 24. The shaded circles show the position of T_{n-1} in T_n when viewed as B. The circles with the X's show the image of T_{n-1} in T_n via g. It is an interesting exercise to redraw the figures to illustrate the dual of Theorem 24 and to find the irredundant chains in Figure 8.

The map f in Theorem 24 is surjective. The inverse image of any element of B is a chain in A. In particular, given $x = (1, x_2, \ldots, x_n)$ in B, $f^{-1}(\{x\}) = \{(q, x_2, \ldots, x_n) \mid \exists k \geqslant 2$ such that $q = k = x_k$ and $x_j \leqslant k$ for $2 \leqslant j \leqslant k\}$.

Theorem 14 proves that every finite lattice can be embedded in some finite extremal lattice. Because of the nice mapping properties of Tamari lattices it would be handy if every finite lattice could be embedded in some finite Tamari lattice. Unfortunately, the answer to this question is negative, since M_5 cannot be embedded in any T_n. [26] showed that T_n is semidistributive; all sublattices of a semidistributive lattice are semidistributive; M_5 is not semidistributive. ■

REMARK 16. Garrett Birkhoff observed that there is an epimorphism from T_4 onto B_3 where B_k is the finite Boolean algebra with k atoms. This is a special case of a more general result proved in the next theorem. ■

THEOREM 25. *B_n is a retract of T_{n+1}.*

Proof. The proof is by induction on n. For $n = 0, 1$ the result is trivial since $B_n = T_{n+1}$. Now suppose that the result is true for $n = k$ and let $\alpha: B_k \to T_{k+1}$ and $\beta: T_{k+1} \to B_k$ be lattice homomorphisms such that $\beta(\alpha(x)) = x$ for all $x \in B_k$. Now B_{k+1} has a coprime/prime decomposition of the type described in Theorem 18. In particular, both A and B in this case are isomorphic to B_k and the decomposition maps f_1 and g_1 are lattice isomorphisms. Think of B_k as B, the lower part, of the coprime/prime decomposition of B_{k+1}.

According to Theorem 24 T_{k+2} has a coprime/prime decomposition such that the decomposition maps f_2 and g_2 are lattice homomorphisms. Furthermore, f_2 is surjective and g_2 is injective. Think of T_{k+1} as B, the lower part, of the coprime/prime decomposition of T_{k+2}.

Define the monomorphism $\sigma: B_{k+1} \to T_{k+2}$ as follows. If $x \in B_k$ then $\sigma(x) = \alpha(x)$. If $x \in B_{k+1} - B_k$ then $\sigma(x) = g_2(\alpha(f_1(x)))$. Now define an epimorphism $\pi: T_{k+2} \to B_{k+1}$ as follows. If $x \in T_{k+1}$, let $\pi(x) = \beta(x)$. If $x \in T_{k+2} - T_{k+1}$, let $\pi(x) = g_1(\beta(f_2(x)))$. The details of showing that σ is a monomorphism and π an epimorphism are routine.

It remains to show that $\pi(\sigma(x)) = x$ for all $x \in B_{k+1}$. If $x \in B_k$, $\pi(\sigma(x)) = \beta(\alpha(x)) = x$. If $x \in B_{k+1} - B_k$, then $\pi(\sigma(x)) = g_1(\beta(f_2(g_2(\alpha(f_1(x)))))) = g_1(\alpha(\beta(f_1(x)))) = g_1(f_1(x)) = x$ since f_1 and g_1 are inverse isomorphisms. ■

From Theorem 25 it follows that B_n is a sublattice of T_{n+1} from which the following Corollary follows.

COROLLARY. *Every distributive lattice of length n is a sublattice of T_{n+1}.* ■

REMARK 17. It is of interest to characterize the elements of T_{n+1} that form a sublattice isomorphic to B_n. Because T_{n+1} is self-dual there are at least two sublattices. I will briefly describe the sublattice that results from applying the recursive construction of Theorem 25. The atoms of the sublattice isomorphic to B_n are all the join-irreducibles of the form $[k, k+1]$ and the coatoms are all the meet-irreducibles of the form $\langle 1, k \rangle$. The elements belonging to the sublattice can best be characterized as the set of all vectors in T_{n+1} such that the consecutive components of the vector are monotonically increasing. Thus, (1, 3, 3, 4) belongs to the sublattice, but (4, 3, 3, 4) does not. The proofs of these statements are left to the reader. ■

Acknowledgements

I would like to acknowledge Garrett Birkhoff for many helpful suggestions and for stimulating the investigation that led to these results. I also would like to thank M. K. Bennett for her many suggestions that improved this paper, and to Curtis Greene for some stimulating suggestions. Finally, I would like to acknowledge the many helpful comments made by the referees, which included a substantial improvement in Figure 6.

References

1. S. P. Avann (1961) Application of the join-irreducible excess function to semi-modular lattices, *Math. Annalen* **142**, 345–354.
2. V. K. Balachandran (1955) On complete lattices and a problem of Birkhoff and Frink, *Proc. AMS* **6**, 548–553.
3. B. Banaschewski (1985) Prime elements from prime ideals, *Order* **2**, 211–213.
4. M. K. Bennett and Garrett Birkhoff (1990) Two families of Newman lattices, preprint.
5. Garrett Birkhoff (1940) *Lattice Theory*, AMS Colloquium Publ., Vol. 25, 1st ed., Providence.
6. Garrett Birkhoff (1967) *Lattice Theory*. AMS Colloquium Publ., Vol. 25, 3rd ed., Providence.
7. Peter Crawley and Robert P. Dilworth (1973) *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, NJ.
8. C. Chameni-Nembua and B. Monjardet (1992) Les trellis pseudocomplémentés finis, *European J. Combinatorics* **13**, 89–107. An English summary of these results is to appear in *Discrete Mathematics*.
9. Paul H. Edelman and Michael E. Saks (1988) Combinatorial representation and convex dimension of convex geometries, *Order* **5**, 23–32.
10. Marcel Erné (1988) Weak distributive laws and their role in lattices of congruences and equational theories, *Algebra Universalis* **25**, 290–321.
11. H. S. Gaskill and J. B. Nation (1981) Join-prime elements in semidistributive lattices, *Algebra Universalis* **12**, 352–359.
12. E. Graczynska (1977) On the sums of double systems of lattices and DS-congruences of lattices, *Coll. Math. Soc. Janos Bolyai* **17**, 161–166.
13. George Grätzer (1978) *General Lattice Theory*, Academic Press, NY.
14. Curtis Greene and George Markowsky (1974) A combinatorial test for local distributivity, Research Report RC5129, IBM T.J. Watson Research Center, Yorktown Heights, NY. The proof of the main result also appears in Markowsky 1980.
15. S. Huang and D. Tamari (1972) A simple proof for the lattice property, *J. Combinatorial Theory* **A13**, 7–13.
16. B. Korte, L. Lovasz, and R. Schrader (1991) *Greedoids, Algorithms and Combinatorics 4*, Springer Verlag, Berlin.
17. George Markowsky (1973) Some combinatorial aspects of lattice theory with applications of the enumeration of free distributive lattices, Ph.D. Thesis, Harvard University.
18. George Markowsky (1973) *Some Combinatorial Aspects of Lattice Theory*, Proc. Univ. of Houston Lattice Theory Conf., Houston 1973, 36–68.
19. George Markowsky (1975) The factorization and representation of lattices, *Transactions of the American Mathematical Society* **203**, 185–200.

20. George Markowsky (1980) The representation of posets and lattices by sets, *Algebra Universalis* **11**, 173–192.
21. George Markowsky and Mario Petrich (1977) Subprojective lattices and projective geometry, *Journal of Algebra* **48**(2), 305–320.
22. Bernard Monjardet (1990) The consequences of Dilworth's work on lattices with unique irreducible decompositions, pp. 192–199 & 201 in *The Dilworth Theorems*, ed. by K. P. Bogart, R. Freese, and J. P. S. Kung, Birkhauser.
23. Ivan Rival (1982) The retract construction, pp. 97–122 in *Ordered Sets*, ed. by Ivan Rival, D. Reidel, Dordrecht.
24. Richard P. Stanley, (1986) *Enumerative Combinatorics, Vol. I.* Wadsworth, Monterey CA.
25. Gabor Szasz (1963) *Introduction to Lattice Theory*. Academic Press, New York.
26. Alasdair Urquhart (1978) A topological representation theory for lattices, *Algebra Universalis* **8**, 45–58.
27. Rudolf Wille (1982) Restructuring lattice theory: an approach based on hierarchies of concepts, pp. 445–470 in *Ordered Sets*, ed. by Ivan Rival, D. Reidel, Dordrecht.