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Representation of Inclines*

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Abstract. In this paper, we show that every incline consisting of Boolean matrices of a given finite size is a subincline of a specific family of such inclines, and consider the question whether every finite incline is embeddable in the semiring of Boolean matrices.

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1 Introduction

Inclines are a generalization of Boolean and fuzzy algebras, and can represent intensity of relationships in a more dynamic way.

Definition 1.1. An *incline* is a set $\mathcal{K}$ on which are defined binary operations, denoted by $+$ and $\cdot$, satisfying the following axioms ($a, b, c \in \mathcal{K}$):

(1) $+$ is commutative: $a + b = b + a$,
(2) $+$ and $\cdot$ are associative: $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$,
(3) $\cdot$ distributes over $+$: $a(b + c) = ab + ac$, $(b + c)a = ba + ca$,
(4) $+$ is idempotent: $a + a = a$,
(5) the incline property holds: $a + ac = a$, $c + ac = c$.

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Thus, an incline is a semiring with idempotent addition, in which the product with a suitable ordering is less than or equal to either factor. Products reduce the value of quantities and make them go down, which is why these structures were named inclines. Inclines were invented by Z. Q. Cao. A good basic reference on inclines is [4]. Note that [4] deals primarily with commutative inclines, i.e., inclines having a commutative product operation. Noncommutative inclines are discussed in [4, Subsec. 3.6].

The three most important inclines are given by the following examples.

Example 1. $\mathcal{K}_1$ is the fuzzy algebra $[0,1]$ under maximum and minimum.

Example 2. $\mathcal{K}_2$ is $[0,1]$ under $\min\{x,y\}$ and $\min\{x+y,1\}$.

Example 3. $\mathcal{K}_3$ is $[0,1]$ under $\max\{x,y\}$ and xy.

Finitely generated inclines have a strong ascending chain condition. Inclines representable in products of linearly ordered inclines must satisfy relations such as $\min\{ax, by\} \le ax + by$ (see [3]).

The incline theory is closely related to the lattice theory. To establish the connection more clearly, we briefly review some definitions from the lattice theory. A *lattice* is a poset in which the supremum and infimum of any finite subset exist. A *semilattice* is a set with a single binary, idempotent, commutative and associative operation. A *finite* semilattice $(S,+)$ with 0 is a lattice under the ordering $y \le x$ if and only if $x+y=x$. In this case, $+$ is supremum and the sum of all elements less than or equal to both of two elements is the infimum of those elements.

A lattice is said to be *distributive* if the operations supremum and infimum satisfy both distributive laws. Every distributive lattice is an incline under supremum and infimum. The additive semigroup of any incline with 0 is a semilattice. Thus, an incline with 0 (any finite incline must have 0 as the product of all its elements) is a semilattice-ordered semigroup. Conversely, every lattice with 0 has a trivial incline structure in which multiplication is 0. Throughout this paper, we identify the supremum operation of lattices with $+$ and the infimum operation of lattices with $\times$.

Binary relations on any set form a semiring under the operations of union and composition. For a set of n elements $\{x_1, \cdots, x_n\}$, this is isomorphic to the semiring of $n \times n$ matrices over a two-element Boolean algebra $\{0,1\}$, where we set $a_{ij} = 1$ or 0 according to whether (x_i, x_j) belongs in the binary relation. Accordingly, these matrices are known as Boolean matrices. We will denote the set of $n \times n$ Boolean matrices by $\mathcal{B}_n$.

Definition 1.2. The *row space* of a Boolean matrix A is the set of all Boolean linear combinations of rows of A. It is also $\{vA : v \in \{0,1\}^n\}$.

The row space of any finite Boolean matrix is a semilattice under addition and therefore is a lattice. Conversely, any finite semilattice can be represented as a semilattice of sets (order ideals) under intersection. By duality, it can also be represented as a semilattice of sets under union, and ultimately as the row space of a Boolean matrix. This situation was studied in detail in [12].

2 Representation of Inclines by Binary Relations and Other Structures

The following question was raised by Schein [15]: *When can an additively idempotent semiring $\mathcal{R}$ be represented as a subring of the semiring of all binary relations?* Schein [16], Andreka [1], and Nemeti [14] have obtained deep results in attempting to answer this question. We would like to pose the related question: *When can a finite incline be represented as an incline of binary relations on a finite set?*

Definition 2.1. A family of finite lattices $\{\mathcal{L}_n\}$ is $(0,1)$-*universal* if every finite lattice embeds as a sublattice in some $\mathcal{L}_n$ by a map preserving 0 and 1.

Theorem 2.2.
(1) *Every finite, additively idempotent semiring can be embedded in the additively idempotent semiring of additive endomorphisms of a finite lattice.*
(2) *Every finite incline embeds in the incline of the semiring of additive endomorphisms of a finite lattice which satisfy $f(x) \le x$. This semiring is an incline.*
(3) *If a finite, additively idempotent semiring is a distributive additive semilattice, then it can be represented as a subsemiring of a semiring of Boolean matrices.*
(4) *Every finite, additively idempotent semiring embeds in the semiring of additive endomorphisms of some member of any $(0,1)$-universal family of lattices.*

Proof. (1) If 1 and 0 are not contained in the existing semiring, we add them. Now the additive semigroup is a lattice. Since it contains 1 and since $+$ is idempotent, the semiring is faithfully represented by the endomorphisms $x \mapsto ax$.

(2) Under the representation used in (1), any incline goes to the maps satisfying $f(x) \le x$. Conversely, if $f(x) \le x$ and $g(x) \le x$, then $f(g(x)) \le g(x) \le x$. Furthermore, since $g(x) \le x$ and f preserves order, $f(g(x)) \le f(x)$. Thus, the decreasing additive homomorphisms form an incline.

(3) Any finite distributive lattice $\mathcal{D}$ is a retract of the semilattice $\mathcal{S}$ of subsets of an n-element set, where n is the number of elements in $\mathcal{D}$ ([2, pp. 58–59]). In other words, there is an inclusion of $\mathcal{D}$ into $\mathcal{S}$ and an epimorphism of $\mathcal{S}$ onto $\mathcal{D}$, whose composition is the identity on $\mathcal{D}$. The inclusion f from $\mathcal{D}$ into $\mathcal{S}$ is given by order ideals, i.e.,

$$f(x) = \{y \in \mathcal{D} : y \le x \text{ and } y \text{ is not a sum of smaller elements and } 0\},$$

and the map g from $\mathcal{S}$ onto $\mathcal{D}$ is given by $g(s) = \inf\{y \in \mathcal{D} : s \subseteq f(y)\}$.

This means that every semilattice endomorphism h of $\mathcal{D}$ gives rise to the semilattice endomorphism of $\mathcal{S}$ that sends s to $f(h(g(s)))$. Since all maps are additive, the correspondence is well defined. Furthermore, for any two semilattice endomorphisms h_1 and h_2 of $\mathcal{D}$ (representing elements of the semiring), $f(h_1(g(f(h_2(g(v)))))) = f(h_1(h_2(g(v))))$ by the retraction property, so the correspondence is a semiring homomorphism.

(4) Let $\mathcal{Q}$ be a finite, additive idempotent semiring. From (1), $\mathcal{Q}$ embeds in the semiring of additive endomorphisms of a finite $\mathcal{M}$. There exists $\mathcal{L}_n$ such that $\mathcal{M}$ can be embedded via a lattice morphism f into $\mathcal{L}_n$ so as to preserve 0 and 1. As in (3), we can define an epimorphism g from $\mathcal{L}_n$ to $\mathcal{M}$ by $g(u) = \inf\{v \in \mathcal{M} : f(v) \geq u\}$. g is well defined if 1 goes to 1 and preserves 0 if 0 goes to 0. Since $\mathcal{M} \subseteq \mathcal{L}_n$ as lattices, the infimum may be taken in either lattice.

If u is in $\mathcal{M}$, it is easy to see that $g(f(u)) = u$ and g is order-preserving. Since f is a lattice morphism and hence, preserves infima, it is easy to see that $f(g(q)) \geq q$. We now claim that $g(u+v) = g(u)+g(v)$. Since g is order-preserving, $g(u+v) \geq g(u), g(v)$, whence $g(u+v) \geq g(u)+g(v)$. On the other hand, $f(g(u)+g(v)) \geq f(g(u)), f(g(v)) \geq u, v$. So, $f(g(u)+g(v)) \geq u+v$, and hence, $g(u)+g(v) \geq g(u+v)$. Thus, $g(u+v) = g(u)+g(v)$ and g is a join-semilattice homomorphism. f is a retract as in (3) and we conclude the argument with the construction used in (3). □

The process for constructing one-sided inverses of retracts that was used in the preceding theorem is a special case of a more general result discussed in [12, Theorem 2.12]. For additional applications of and references on retracts, see [13].

Any finite lattice is the row space of a Boolean matrix (cf. [6]). Given a finite lattice $\mathcal{L}$ which is isomorphic to the row space of the Boolean matrix $\mathcal{M}$, every additive endomorphism h of $\mathcal{L}$ can be represented by left multiplication of $\mathcal{M}$ by a matrix $\mathcal{A}$. In general, $\mathcal{A}$ is not unique. If $\mathcal{M}$ is a $k \times q$ matrix, then $\mathcal{A}$ is a $k \times k$ matrix, and the product $\mathcal{AM}$ has as its rows the images by h of the rows of $\mathcal{M}$. For any given $\mathcal{M}$ and h, there is a unique maximal $\mathcal{A}$ in the sense that it has the largest possible number of ones. We denote this maximal matrix by $\mathcal{P}(h)$ and we have that $\mathcal{P}(h)_{ij} = 1$ if and only if $\mathcal{M}_{j*} \leq h(\mathcal{M}_{i*})$. It follows easily from the definition that $\mathcal{P}(h)\mathcal{P}(g) \leq \mathcal{P}(hg)$ and $\mathcal{P}(h) + \mathcal{P}(g) \leq \mathcal{P}(h+g)$. We do not have equality for arbitrary lattices.

Example 4. Consider the lattice Λ_n with additive generators (join-irreducibles) x_i $(i = 1, \cdots, n)$ such that $x_i + x_j = 1$.

Λ_n is isomorphic to the row space of the matrix $\mathcal{M}_n$ which is the complement of the $n \times n$ identity matrix $\mathcal{I}_n$. The automorphisms of Λ_n correspond exactly to the permutations of the x_i. For each automorphism τ, $\mathcal{P}(\tau)$ is just the corresponding permutation matrix.

Given any two distinct automorphisms τ_1 and τ_2, their sum will be the endomorphism that sends every nonzero element of Λ_n to 1. However, the sum $\mathcal{P}(\tau_1) + \mathcal{P}(\tau_2)$ will vary. Thus, $\mathcal{P}$ is not additive.

Proposition 2.3. *For $n \geq 3$, the semiring of lattice endomorphisms of Λ_n will not embed in the semiring $\mathcal{B}_m$ for any m.*

Proof. Assume to the contrary that there is such an embedding of $\mathrm{End}(\Lambda_n)$ in $\mathcal{B}_m$ for some m and n. The permutations embed in a single $\mathcal{H}$-class containing the identity matrix $\mathcal{I}_m$ and the permutation group. The permutations act faithfully as permutations of a row basis of $\mathcal{I}_m$, preserving its order structure.

Consider the case where $n = 3$. A faithful permutation representation will have some orbit equivalent either to the standard representation or to the regular representation (consider possible isotropy groups for a faithful representation). Then the row basis elements cannot have any element greater than another.

This gives a block in the matrix $\mathcal{I}_m$ where the permutations are represented either by the standard representation or by the regular representation. Let τ_1, τ_2, τ_3 be the identity and two 2-cycles, respectively. On the lattice Λ_n, $\tau_1 + \tau_2 = \tau_2 + \tau_3$. But, in this block of $\mathcal{I}_m$, we cannot have the equality of $(\tau_1 + \tau_2)\mathcal{I}_m$ and $(\tau_2 + \tau_3)\mathcal{I}_m$ which are equal to $\tau_1 + \tau_2$ and $\tau_2 + \tau_3$, respectively. □

Definition 2.4. Let Q be a quasiorder on an n-element set represented in $\mathcal{B}_n$ by an idempotent, reflexive Boolean matrix E. The incline $\mathcal{K}(Q)$ is the subsemiring of matrices M such that $M = EME$ and $M \leq E$.

It is easy to see that $\mathcal{K}(Q)$ is closed under multiplication and addition, and it is therefore a semiring. The incline property follows from $MN \leq ME = M$ and $NM \leq EM = M$. As defined above, it might appear that $\mathcal{K}(Q)$ depends on the matrix used to represent Q. The following theorem shows that this is not the case and the definition is independent of the representation used.

Theorem 2.5. *Every finite incline which is a subsemiring of $\mathcal{B}_n$ is a subsemiring of $\mathcal{K}(Q)$ for some quasiorder Q. Isomorphic quasiorders give isomorphic semirings, and $\mathcal{K}(Q)$ is also unchanged if each equivalence class of Q is replaced by a single point.*

Proof. Let Ω be the sum of all elements of the incline. Then $\Omega\Omega \leq \Omega$ and ΩG, $G\Omega \leq G$ for any G in the incline by the incline property. Moreover, $G \leq \Omega$. Now let $E = I + \Omega$, where I is the identity matrix. This proves the first statement.

The indifference relation $\mathcal{D}$ of E, which is the intersection of E and its transpose, lies between I and E. Hence, $\mathcal{DMD} = M$ for $M \in \mathcal{K}(Q)$. Thus, if we block all matrices according to equivalence classes of $\mathcal{D}$, then every block consists entirely of zeroes or ones. Replacing all such blocks by 0 or 1 is a semiring isomorphism. The structure is also unchanged if we conjugate by any permutation matrix.

Another way of viewing $\mathcal{K}(Q)$ is in terms of the relations represented by the matrices. These are subrelations R of Q having the property that, if $(a,b) \in R$, then $(x,y) \in R$ for all x and y such that $x \leq a$ and $b \leq y$, where x and y are in the domain of Q. □

Example 5. If we take the relation $i < j$ of integers, the maximal incline corresponding to this is the incline of all subtriangular matrices M such that, if $m_{ab} = 1$, $x \leq a$, and $b \leq y$, then $m_{xy} = 1$.

Theorem 2.6.

(1) *If P_0 is a subposet of P, then $\mathcal{K}(P_0)$ is a subsemiring of $\mathcal{K}(P)$.*

(2) *Suppose P is a poset with n elements. Let Z_n be the poset of all subsets of an n-element set. Then $\mathcal{K}(P)$ is isomorphic to a subincline of $\mathcal{K}(Z_n)$.*

Proof. As noted in Theorem 2.5, $\mathcal{K}(P)$ consists of all matrices corresponding to binary relations R contained in the order relation of P such that, if $(a,b) \in P$, $(b,c) \in R$, and $(c,d) \in P$, then $(a,d) \in R$.

To simplify the exposition, we identify P_0 with a subset of P, represent $\mathcal{K}(P)$ using a matrix E, and represent $\mathcal{K}(P_0)$ using a matrix E_0 such that $E_0 \leq E$ and $EE_0E = E_0$. Now, given M such that $E_0ME_0 = M$, it follows that $EME = EE_0ME_0E = E_0ME_0 = M$. It is now easy to see that the first part of the theorem holds.

To see that the second part holds, note that every poset is a subposet of its power set. Simply select the set of principal order ideals and note that P is isomorphic to this set ordered by set inclusion. □

Our general problem can now be restated: *Does the incline of decreasing endomorphisms of an arbitrary semilattice for some n embed in the incline $\mathcal{K}(Z_n)$ where Z_n is the poset of all subsets of an n-element set?* One could approach this by an induction, assuming that the incline has a unique minimal nonzero element y and that we have a representation for the quotient of the incline by the ideal $[0,y]$.

Here is another interesting question: *Is there a simple $(0,1)$-universal family of lattices that is easy to describe, but which is not the family of all finite lattices?* Alternatively, this can be relaxed a little for our purposes to the following question: *Is there a family of finite semilattices such that every finite semilattice is a retract of some member of this family by a map preserving sums and 0?*

Yet another question whose answer is unknown to us is: *What are all the finite, simple additively idempotent semirings?* Matrices over finite fields are one example, and another is the semiring of Boolean matrices. To see that the last entity qualifies as an example, note that any 2-sided ideal must contain a nonzero rank 1 matrix. Therefore, it contains every rank 1 matrix, whence it contains every matrix.

Proposition 2.7.

(1) *The only finite simple inclines are those of 1 and 2 elements. Every other incline has an ideal which is also an order ideal.*

(2) *Every incline with at least 3 elements has a 2-sided ideal which is also an order ideal such that the quotient semiring has 2 elements.*

Proof. If L is an incline which has at least 3 elements, there exists a minimal nonzero element x which is not 1. Then $[0,x]$ is a 2-sided ideal and an order ideal. The last statement follows by induction and the fact that the inverse image of an order ideal is also an order ideal. □

The preceding proposition could also be useful in studying representations.

3 Other Algebraic Properties of Inclines

Definition 3.1. The *n-dimensional vectors* over an incline $\mathcal{K}$ are the elements of the n-fold Cartesian product $\mathcal{K}^n$. The *n-square matrices* over an incline are all $n \times n$ arrays (a_{ij}) of elements from the incline added and multiplied as usual with the componentwise operations being performed in the incline.

A *subspace* of the n-dimensional vectors is a subset that is closed under sum and scalar product. A *basis* of a subspace is a linearly independent set of vectors whose linear combinations generate the subspace. A *standard basis* is a basis $\{x_1, \cdots, x_n\}$ such that $x_i = c_{ii}x_i$ whenever $x_i = \sum c_{ij}x_j$.

The following theorem is due to Cao [3].

Theorem 3.2. *Every finite incline in which idempotents are linearly ordered has a unique standard basis.*

Example 6. The 4-element Boolean algebra does not have a unique standard basis.

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ are, respectively, equivalent to the equality of row spaces, equality of column spaces, and equality of both row and column spaces. Two matrices A and B are $\mathcal{D}$-equivalent if and only if there is an isomorphism between their row spaces, or equivalently, if there is an isomorphism between their column spaces.

Idempotent Boolean matrices can be reduced to a very special triangular form in which nonzero rows and columns are idempotent (cf. [6, 11]).

Idempotent matrices E over a finite incline can be reduced to a standard form F such that $F \leq E$. They have the same row space, the nonzero rows of F are a standard basis, and $f_{ii}F_{i*} = F_{i*}$ for all i where i denotes the ith row of F. In particular, diagonal elements are then idempotent.

If the idempotents in this incline are linearly ordered, it is possible to have F triangular in the weaker sense that the binary relation $\{(i,j) : m_{ij}^{n-1} > m_{ji}\}$ is triangular.

Example 7. The following matrix is idempotent when considered over the incline $\mathcal{K}_3$ discussed at the beginning of this paper.

$$\begin{pmatrix} 1 & 0.49 & 0.7 \\ 0.7 & 1 & 0.49 \\ 0.49 & 0.7 & 1 \end{pmatrix}$$

Proposition 3.3. *For any matrix M over a finitely generated incline, the sum $I + M + M^2 + \cdots$ converges and yields an idempotent.*

Proof. The convergence follows from the ascending chain condition in finitely generated inclines. In this case, the sum equals its own square because of additive idempotence. □

A *topological incline* is an incline which is a topological space and such that the operations are continuous. $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are topological and can be characterized by their special properties on $[0,1]$. $\mathcal{K}_1$ is characterized by all elements being idempotent. $\mathcal{K}_2$ is characterized by 0 and 1 being the only idempotents and every other element having a finite power which is 1. Note that in $\mathcal{K}_2$, $1 < 0$ in the induced ordering. $\mathcal{K}_3$ is characterized by 0 and 1 being the only idempotents and having no other elements with a finite order.

Any topological incline on $[0,1]$ can be constructed by inserting copies of $\mathcal{K}_2$ and $\mathcal{K}_3$ into subintervals of $\mathcal{K}_1$. One may also study infinite-dimensional matrices over a topological incline. For instance, it can be shown that over $\mathcal{K}_3$ any symmetric infinite-dimensional matrix with a largest entry has a nonzero eigenvalue.

The 2-sided ideals in any semiring with 0 (or in any semigroup) form an incline (noncommutative in general) under the operations $\mathcal{I}+\mathcal{J}$ and $\mathcal{I}\mathcal{J}$. In [10], we determined these semirings for algebraic number rings, polynomial rings in one variable, and regular semigroups. For nilpotent semigroups, the incline structure of the incline of ideals determines the semigroup structure (see [10]).

There is another incline very closely related to the structure of semirings, the incline of topologizing filters (see [5]).

Definition 3.4. A *topologizing filter* on a semiring $\mathcal{R}$ is a collection Δ of left ideals such that

(1) if $\mathcal{I} \in \Delta$ and $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{J} \in \Delta$;
(2) if $\mathcal{I}, \mathcal{J} \in \Delta$, then $\mathcal{I} \cap \mathcal{J} \in \Delta$;
(3) if $\mathcal{I} \in \Delta$, $r \in \mathcal{R}$, then $(\mathcal{I} : r) \in \Delta$, where $(\mathcal{I} : r) = \{x \in \mathcal{R} : xr \in \mathcal{I}\}$.

Example 8. Every topologizing filter for $\mathcal{Z}$ consists of the ideals containing some given ideal (m).

Definition 3.5. The *product* $\mathcal{F}_1\mathcal{F}_2$ of two topologizing filters is the set of left ideals

$$\{\mathcal{I} : \exists \mathcal{H} \in \mathcal{F}_1,\ \mathcal{H} \supseteq \mathcal{I},\ (\mathcal{I} : h) \in \mathcal{F}_2\ \forall h \in \mathcal{H}\}.$$

Proposition 3.6. [10] *For any semiring with 0, the set of topologizing filters forms an incline under the operations of intersection and product.*

The previous proposition generalizes the earlier result for topologizing filters on rings.

For semirings, congruences can be as important as ideals. The natural generalization of the previous definition seems to us to be the following definition.

Definition 3.7. A *topologizing filter of left congruences* on a semiring $\mathcal{R}$ is a set of left congruences L_c (taken as binary relations on $\mathcal{R}$) such that

(1) if $C \in L_c$ and $C \subseteq D$, then $D \in L_c$;
(2) L_c contains the intersection of any two of its elements;
(3) if $C \in L_c$ and $x, y \in \mathcal{R}$, then $\{C : (x,y)\} \in L_c$, where $\{C : (x,y)\}$ is the congruence generated by all unordered pairs $r, s \in \mathcal{R}$ such that $(rx + sy, ry + sx) \in C$.

Definition 3.8. The *product* $\mathcal{F}_1\mathcal{F}_2$ of two topologizing filters of left congruences is the set of all congruences Ψ such that, for some $\Psi_1 \in \mathcal{F}_1$, Ψ_1 contains Ψ and, for all $(x,y) \in \Psi_1$, $\{\Psi : (x,y)\} \subseteq \mathcal{F}_2$.

This raises the following basic question: *Are there any conditions under which the topologizing filters of left congruences form an incline for semirings that are not rings?*

The following concepts are important in symbolic dynamics.

Definition 3.9. Matrices A and B are *shift equivalent of lag n* over a semiring $\mathcal{R}$, if there exist matrices R and S over $\mathcal{R}$ and a nonnegative integer n such that $RA = BR$, $AS = SB$, $RS = B^n$, and $SR = A^n$. The *strong shift equivalence* is the transitive closure of lag 1 shift equivalence.

Another problem which would be of interest is the strong shift equivalence of matrices over inclines. The Boolean case was settled in [7, 8]. Kim and Roush [9] describe the shift equivalence for matrices over inclines.

A final question that we wish to raise is: *Are two matrices over a finite incline strong shift equivalent if they have the same trace and are shift equivalent?*

4 Conclusion

Various families of finite inclines and of finite inclines representable by Boolean matrices have been discussed. Inclines play an important role in the algebraic structure of general rings and semirings.

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