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NUMERICAL TIC-TAC-TOE – I

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ABSTRACT

This article discusses two very interesting variations of Tic-Tac-Toe that use numbers instead of X's and O's. These variations were discovered by Ron Graham and P. H. Nygaard. Analyses of both games are presented along with several open problems that can be fruitfully explored at many different levels. Part II presents a complete winning strategy for Graham's version. The article also presents the history of these games to the extent that the author was able to determine it.

Introduction

Tic-Tac-Toe is a game that is known to everyone. It has the great virtue of being easy to learn and requiring almost no equipment to play. Unfortunately, it quickly becomes boring to play since players realize that any game played with moderate care should end in a tie. This article describes some variations of Tic-Tac-Toe due to Ron Graham and P. H. Nygaard that share the virtues of Tic-Tac-Toe, but which are challening enough to provide hours of play.

The Graham and Nygaard variations of Tic-Tac-Toe are simple enough so that grade school children can play them, but are a good source of open problems that seem to have been explored very little. I hope that this article will lead to both of these games being better known and to this area of recreational mathematics being better explored.

After describing the Graham and Nygaard games, I will give as much of their history as I have been able to find. I am interested in receiving additional information about these or related games to supplement what I have been able to uncover. Following the history of the games, I will describe some computer analyses of the games. The article concludes with a discussion of open problems.

Ron Graham's Game

Ron Graham's Game uses the standard 3x3 Tic-Tac-Toe board. Instead of calling the players X and O, we call them Odd and Even. Odd gets the numbers 1, 3, 5, 7, and 9, while Even gets the numbers 2, 4, 6, and 8. Odd goes first, after which the players take turns placing one of their numbers in the empty cells of the standard Tic-Tac-Toe board. Numbers may be used only once. The object of the game is to be the player who completes a line that sums to 15. As in Tic-Tac-Toe, a line is a row, column, or diagonal.

Completing a line means putting the final number in the line so it sums to 15. Players are allowed to use numbers placed by the opponent to reach the sum of 15. Once a line contains two numbers whose sum is 15 or greater there is no way to complete that line, although filling in the remaining cell might be necessary to complete a different line.

The setup for playing Graham's Game is quite simple. Start out with a piece of paper which has the familiar Tic-Tac-Toe board on it. To one side write a row with the numbers 1 3 5 7 9 in it and below it write a row with the numbers 2 4 6 8 in it. As the players use the numbers they must cross them off. Figure 1 shows the setup for playing Graham's Game.

Before you read the analyses of Graham's Game, I recommend that you play it several times so you can form your own opinion about the best strategy.

P. H. Nygaard's Game

P. H. Nygaard's Game is very much like Graham's Game. It uses the same board and numbers, and the players Odd and Even play as before. The only difference is that a player wins by either completing a line of 15 or by getting three numbers of the same parity (odd or even) in a line. In other words, Odd can win by either completing a line that sums to 15 or getting three odd numbers in a line. Similarly, Even can win by either completing a line that sums to 15 or by getting three even numbers in a line. Thus, Nygaard's Game is like playing Tic-Tac-Toe and Graham's Game simultaneously. Figure 2 shows the setup

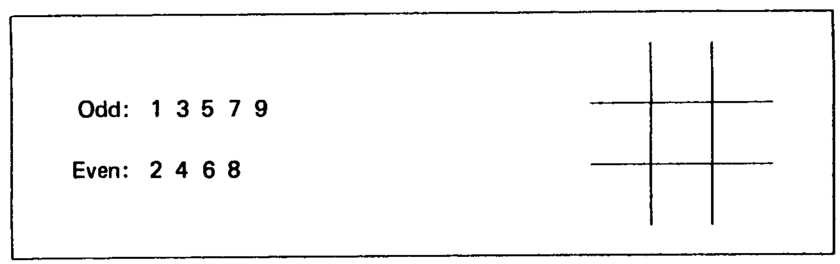

Figure 1. The setup for playing Graham's Game.

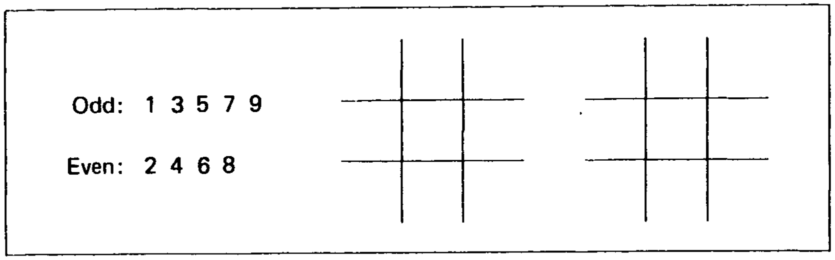

Figure 2. The setup for playing Nygaard's Game.

necessary to play Nygaard's Game. Figure 2 shows two Tic-Tac-Toe boards. One is used to record the numbers and the other one keeps track of the parities and looks like a standard Tic-Tac-Toe game. Apparently, P. H. Nygaard created colored and numbered markers for playing his game that allowed the players to easily see when three numbers of the same parity were in a line. With such markers only one board is required.

As with Graham's Game, I suggest that you play Nygaard's Game several times before reading its analysis. This will allow you to form your own opinion about strategy.

History of the Games

This section presents the history of these two games such as I have been able to determine it and assigns credit in what I hope is a fair manner. I welcome any additional information on the material presented here.

I first learned of Graham's Game from Larry Carter of IBM Research who mentioned it when he was presenting a seminar at the University of Maine. At that time, he stated that the game was invented by Ron Graham, but that it was not known whether either of the players had a winning strategy. Since the game seemed to lend itself to computer analysis, I decided to write a program to analyze the game. After I had written the computer programs described in the next section to analyze the game, I wrote to Ron Graham with my results asking for information. Ron Graham responded by a letter containing the following two paragraphs [3]:

> I'm impressed that you were able to trace the numeric tic-tac-toe game back to me. I first discovered it in the late '50's and I convinced myself that it was (barely) a first player win. All the analysis was done by hand then (it took a few hundred pages) so I was never completely sure of the answer.
>
> I'm not aware of the game ever occurring in the literature before but of course that doesn't mean it isn't there. Martin Gardner would be a good source for this I imagine. I also never found an effective description of a winning strategy (which makes it a nice game to play).

Enclosed with the letter were two sheets from an unpublished manuscript that Ron Graham produced in the 1950's [4]. The pages are covered with trees showing how the game progresses from position to position.

The conclusion Ron Graham mentions in his letter is supported by the analysis I performed and which is described in the next section. Because I did the analysis using two different programs in two very different programming languages, I feel reasonably sure that the final result is correct. Furthermore, the computer analysis produced a "compact" strategy that can be verified by hand by some highly motivated person.

Following the suggestion Ron Graham made in his letter, I wrote to Martin Gardner. Gardner replied by letter. [2]. He did not know of Graham's connection with the game, but had corresponded with P. H. Nygaard about a similar game. Gardner enclosed copies of a letter and an article written by Nygaard.

The article by Nygaard describes Graham's Game, but does not claim that it is his own invention [5]. The letter to Gardner was written in 1965 and claims what I have named Nygaard's Game as his own invention [6]. Figure 3 is a reproduction (the original has extraneous marks on it) of the instructions that Nygaard sent Gardner with his letter [7].

I learned from Helen Nygaard of Spokane, Washington (she is not a relative of P. N. Nygaard) that P. H. Nygaard died in 1976. If anyone has any additional information about Nygaard's experience with his version of Tic-Tac-Toe, I would appreciate hearing from them.

Analyzing the Games

The first step in analyzing these variants of Tic-Tac-Toe is to compute an upper bound for the number of different configurations that could possibly result from playing. The formula for the number of configurations which *N* non-blanks is given by

$$\mathrm{BIN}(9,N) \times \mathrm{BIN}(5,\mathrm{CEIL}(N/2)) \times \mathrm{BIN}(4,\mathrm{FLOOR}(N/2)) \times N!,$$

where *BIN* is the familiar binomial coefficient, *CEIL* is the least integer greater than or equal, and *FLOOR* is the greatest integer less than or equal.

To see that the formula is correct, note that it represents first choosing *N* non-blanks from the 9 cells of the board. Then, CEIL(N/2) odd numbers are selected to put on the board. Following this, FLOOR(N/2) even numbers are selected to put on the board. Finally, the *N* numbers selected are placed into the *N* cells in one of N! ways.

The formula just given is an upper bound on the number of configurations that might have to be analyzed. Not every configuration given can be achieved in legal play. For example, any configuration which has two non-intersecting lines that sum to 15 cannot arise legally since play must stop the instant the first line

ODD AND EVEN

"Odd and Even" is a game for two players utilizing the same three-by-three grid used in "Tic-Tac-Toe." It is played with nine counters made from cardboard or plastic. Five are colored red and four are blue. On the red markers are inscribed the odd numbers 1, 3, 5, 7, and 9. The blue markers carry the even numbers 2, 4, 6, and 8. One player selects the red counters and the other player has the blue counters at his disposal. All markers are left in plain view. The player with the red markers starts by placing one of his markers in one of the nine spaces of the grid. The blue player then puts one of his markers in any vacant space in the grid. The players alternate until the game is over. For successive games the red and blue counters are rotated from one player to the other to equalize any advantage, if any, arising from possession of the red or blue counters.

The game is won by the player who first succeeds in placing his marker so that three markers in a line have the same color or so that three counters in a line add up to a total of 15. Note that two counters in a line, such as 8 and 7, will not win; there must be three in a line. The lines may be vertical, horizontal, or diagonal.

To illustrate, if a line has a red 7 and a blue 2, and the next player has the blue markers, he can win by placing a 6 in a vacant space in this line, because 7 + 2 + 6 = 15. If a line has a red 7 and a red 1, and the next player has the red markers, he can win by placing a red 3 in a vacant space on this line, because now all three markers in this line are red. Players must therefore be alert to two possibilities, three like colors in a line or a total of 15 for three numbers in a line.

The game is much more complicated than "Tic-Tac-Toe." Sometimes neither side wins, but more often the blue or the red player will be successful. A complete analysis of the play of the game is possible, but quite difficult.

Figure 3. P. H. Nygaard's instructions for playing his Game.

of 15 is completed. For our purposes, an upper bound is sufficient. Table 1 shows the values produced by the formula for $N = 1, 2, 3, \ldots, 9$. Altogether, there are 9,335,565 configurations. For comparison purposes, an upper bound on Tic-Tac-Toe configurations is also presented. The large difference in the numbers illustrates why the games we are discussing are a good deal more complicated than Tic-Tac-Toe.

The total number of configurations is a reasonable number for analysis with a computer if a little care is used. The following observations can be used to reduce the number of cases that need to be examined. First, by symmetry, the opening move by Odd can be restricted to the positions shown in Figure 4, which reduces the number of first moves from 5 x 9 = 45 to 5 x 3 = 15.

Table 1. The Number of Configurations in Graham's and Nygaard's Games and in Tic-Tac-Toe

Size	*Upper Bound on Graham/Nygaard Configurations*	*Size*	*Upper Bound on Tic-Tac-Toe Configurations*
1	45	1	9
2	1440	2	72
3	20160	3	252
4	181440	4	756
5	907200	5	1260
6	2419200	6	1680
7	3628800	7	1260
8	1814880	8	630
9	362880	9	126

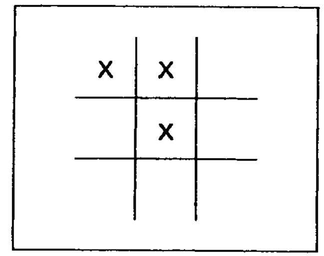

Figure 4.

Another simple, but useful, observation is that if $a, b, c, d, \ldots$, represents a sequence of moves, then the sequence $10 - a, 10 - b, 10 - c, 10 - d, \ldots$, produces exactly the same outcome. To see this, note that $a + b + c = 15$ if and only if $(10 - a) + (10 - b) + (10 - c) = 15$. Furthermore, since $10 - a$ and a have the same parity, three numbers of the same parity are in a line in one game if and only if three numbers of the same parity are in a line in the other game. From this observation it follows that we may restrict the first move to the odd numbers 1, 3, or 5. This reduces the number of first moves that must be examined to 9 instead of the original 45.

The Analysis of Graham's Game

To solve Graham's Game, I wrote a Pascal search program. The program used various pruning tricks extending the tricks already discussed. To avoid recursive explosion, I used dynamic programming which involves storing results for future use so they don't have to be constantly recomputed. To store all the possible game configurations that are counted in Table 1 would take quite a bit of memory. The exact amount would depend on the amount of memory used to store each configuration. Since there are 9335565 configurations, storing all of them would task most computer systems. To obtain good results from dynamic programming, it is only necessary to store those configurations for which recalculation takes much more time than looking the result up in a table. In my program, I stored only the configurations through size 7.

To avoid complicated hashing schemes, I wrote a program that used roughly sixteen megabytes of space when loaded. Because of this extravagant use of space the program finished on an IBM 3090 in several minutes, which was faster than I expected. The results confirmed Ron Graham's belief that Odd has a winning strategy.

The tree searching used by the program is quite standard in the analysis of games. It is based on the fact that the current player wins if some move leads to a forced win for that player. On the other hand, the current player loses if all moves lead to forced wins for the other player. Finally, a game is a tie if all moves lead to configurations from which the other player can force a tie, or if no further moves are possible for either player. These definitions are used recursively.

Furthermore, the program searches to see whether the current player has a one-step win and whether the opponent has any wins. Overall, the combination of all the techniques mentioned above leads to a tremendous reduction in the number of games that must be examined. Table 2 compares the number of configurations that were examined to find a winning strategy with the upper bound on the total number.

Even the reduced number of configurations examined by the Pascal search program is still too large to manipulate manually, so another computer program was needed to verify the results. To verify the results and to produce a compact strategy, I used APL running on an IBM PC. I chose APL because it is quite

Table 2. The Pruning Results for Graham's Game

Size	*Configurations*	*Examined*	*Percent*
1	45	4	9
2	1440	128	9
3	20160	577	3
4	181440	4856	3
5	907200	15247	2
6	2419200	52533	2
7	3628800	115817	3

Table 3. The Pruning Results for Nygaard's Game

Size	*Configurations*	*Examined*	*Percent*
1	45	9	20.0
2	1440	195	14.0
3	20160	470	2.0
4	181440	2639	1.0
5	907200	4141	0.5
6	2419200	7799	0.3
7	3628800	11723	0.3

different from Pascal, and it allows the manual manipulation of large amounts of data. A complete description of a winning strategy for Odd is given in Part II of this article.

The Analysis of Nygaard's Game

By slightly modifying the search program used for Graham's Game, I was able to use it for analyzing Nugaard's Game. The search produced the result that Nygaard's Game should be a tie. Table 3 shows the pruning results achieved by the modified Pascal program. The most startling fact revealed by the table is the sharp reduction in the number of cases that the program examined. It appears that as more numbers are placed on the board, the game begins to resemble conventional Tic-Tac-Toe much more than Graham's Game does.

I verified the results using an APL program as before. Furthermore, the small number of cases listed in Table 3 suggested that the entire game could be analyzed on a standard IBM PC. I wrote a searching program in Turbo Pascal that did an exhaustive search using a compact dynamic table and which produced the same results after running for several hours. The APL program also did a complete search in several hours running on an IBM PS/2 Model 50.

There is an interesting twist to Nygaard's Game which emerged from the analysis: **putting a 5 anywhere on the board as a first move is a losing move for Odd.** It is easy to see why this is so. First, note that the triplets (5,2,8) and (5,4,6) sum to 15. Second, regardless of where Odd puts 5 on the board, Even can put the number 2 next to the 5 so as to threaten to win on the next move. This requires Odd to block Even's threat, which allows Even to use the number 4 to create a dual threat. The sequence of games shown in Table 5 illustrates this point.

After Odd puts a 5 in position *b*, Even puts a 2 in position *a*. This forces Odd to put some odd number *X* in position *c*. Even responds by putting 4 in position *e*. This gives Even two threats: Even can win by putting 6 in either the bottom,

	5	

2	5	

2	5	X

2	5	X
	4	

Figure 5.

middle, or right position. In the first case the second column will sum to 15, while in the second case the diagonal will contain three even numbers. Odd can block at most one of these moves.

It is easy to see that Even has a similar strategy regardless of where Odd places the 5. It is important that Even use one number from each of the two triples mentioned above. Otherwise, Even will not be able to generate this double threat.

Open Problems

Despite the results presented above, there are still many questions that need to be answered about Graham's and Nygaard's games, as well as about other variations of Tic-Tac-Toe. The first question that comes to mind is to produce compact non-table-lookup strategies for Graham's and Nygaard's games. The results in the sections on the analysis of Graham's and Nygaard's games suggest that this more likely to be easier for Nygaard's Game than for Graham's Game.

Another interesting problem is produce complete strategies for both games; namely, an exhaustive list of all winning strategies.

The results here suggest that there may be yet other interesting variations of Tic-Tac-Toe, some of these might turn out to be equivalent to Tic-Tac-Toe. Chapter 22 of *Winning Ways* lists several games that appear to be different but which reduce to conventional Tic-Tac-Toe [7].

Another variation of the games presented here is to use the numbers 1 through 9, but let either play use any remaining number. Players win as in Graham's Game by producing a line that sums to 15. This game is mathematically less interesting than Graham's or Nygaard's games because one player has a very simple strategy. I leave it to you to figure out what this strategy is.

It would also be interesting to have a more extensive bibliography of Tic-Tac-Toe like games and a more complete history than presented here. For example, how far did Nygaard get with his game? Did he produce many sets of markers to play his game in the manner he describes in his letter?

Another interesting problem is to investigate variations of the game where players are required to produce some sum other than 15. How does the strategy

differ if the desired sum is 13 or 14? Alternatively, instead of using the numbers 1 through 9, the players might use the numbers 2 through 10 so Even goes first. This leads to the question of what is the proper sum to use for this game.

Generalizing the ideas in the previous paragraph, further leads to games in which one player gets five integers while the other player gets four. What are interesting numbers to use for such a game and what are the sums that make the game most interesting? Perhaps there are interesting variations that use primes, squares, or other interesting types of integers. Maybe some other operations such as multiplication can also produce interesting games.

What happens on larger grids such as 4x4 and 5x5? Are 3-dimensional variations of these games interesting?

Besides these questions, there are the problems of determining the number of legal configurations that can be produced when playing the game. The numbers in Table 1 are upper bounds on the number of possible legal configurations. While all of the configurations counted in Table 1 are combinatorially possible, some of them cannot result from playing the game because they include more than one win. Before you tackle this problem, I suggest you warm up by calculating the number of legal Tic-Tac-Toe configurations.

Numerical variations of Tic-Tac-Toe seem to be a little-explored area of recreational mathematics that offers significant challenges. Many of the problems could be tackled by high school students and undergraduates and could be useful for giving them a taste of mathematical research. I would be interested in hearing about any results people achieve in this area.

Acknowledgments:

I would like to thank Larry Carter for introducing me to this problem; Gerry Dube and Walt Horbert for their assistance with running my program on the IBM 3090; and Ron Graham, Martin Gardner, and Helen Nygaard for the information they supplied. Finally, I would like to thank Ken Brownstein and Grattan Murphy for some interesting discussions on the subject matter of these games.

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